Fredholm Operators and Spectral Flow
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Dr. Nils Waterstraat
Lecturer (Assistant Professor) in Mathematics (permanent) at the university of Kent

Abstract
Fredholm operators are one of the most important classes of linear maps in mathematics. They were introduced around 1900 in the study of integral operators and by definition they share many properties with linear maps between finite dimensional spaces (i.e. matrices). A further essential class of linear maps are selfadjoint operators, which play a decisive role in mathematics but are also of crucial interest in physics where they represent observables in quantum mechanics. The eigenvalues of a selfadjoint operator are always real, and in the latter picture they are the possible outcomes of a measurement of the observable.
Selfadjoint Fredholm operators have the nice property that, in a neighbourhood of $0 \in R$, there are only finitely many eigenvalues which also all have finite dimensional eigenspaces. Hence, the spectral theory for these eigenvalues is as for a matrix. Atiyah, Patodi and Singer considered in 1976 selfadjoint Fredholm operators depending on a parameter and investigated how such eigenvalues change under variation of the parameter [1]. They introduced an integer, called the spectral flow, which roughly speaking is the number of eigenvalues that cross 0 whilst the parameter travels along its interval of definition. Since then, the spectral flow has reappeared in various different fields of mathematics and its interfaces to physics, e.g. particle physics [9], [5], symplectic analysis [3], [8] and geometry [4], [6], to mention only a few. Also, the spectral flow has been applied in bifurcation theory [2], [7] (see abstract for the summer school). We have used it in about 15 publications in different interfaces of all previously mentioned subjects.
The aim of this lecture series is to give an elementary introduction to the spectral flow partially following the lecture notes [11], which came out of a PhD course at the University of Turin in 2012 and were recently updated and revised. The schedule for the lectures is as follows:

Lecture 1 (3h): Bounded and Unbounded Operators
We recap the definitions of bounded operators and their norms on Hilbert spaces. As particularly important examples, we recall projections and compact operators. Afterwards, we consider densely defined operators and introduce closedness and the adjoint of an operator.

Lecture 2 (3h): Symmetric, Selfadjoint and Fredholm Operators
In the first half of this lecture, we introduce the graph norm of an operator and define Fredholm operators. In the second half, we introduce symmetric and selfadjoint operators and discuss their difference. Finally, we briefly speak about functional calculus and define spectral projections.

Lecture 3 (3h): Topologies on the Space of Selfadjoint Fredholm Operators
For speaking about paths of operators, we need to define what this should mean, i.e. we need a metric on the set of selfadjoint Fredholm operators. If we only consider bounded operators, we always get a canonical metric from the operator norm. However, things are less clear when we allow our operators to be unbounded. Here we introduce the gap metric and the Riesz metric and report about their relation. Also, we discuss some topological properties of the space of selfadjoint Fredholm operators following classical work of Atiyah and Singer as well as more recent work by Lesch.

Lecture 4 (3h): The Spectral Flow - Definition and Properties
The aim of this lecture is to define the spectral flow for paths of selfadjoint Fredholm operators that are continuous with respect to the gap metric. Given the preliminaries from the previous lectures, the exact definition is now surprisingly simple. Roughly speaking, we only need a little spectral stability from Lecture 3 and the spectral projections from Lecture 2. We then discuss the basic properties of the spectral flow and, as highlight, its homotopy invariance which is a rather strong property. If time allows, we make some comments to what extent these properties uniquely characterise the spectral flow.

Lecture 5 (3h): Examples and Crossing Forms
We begin this lecture by an elementary example. Then we address the question how the spectral flow can be computed for more complicated paths. We introduce the so called crossing forms that were introduced by Robbin and Salamon in their seminal work [8] and recently generalized by us in [10].

References

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