ELEMENTARY MATHEMATICS

W W L CHEN and X T DUONG

Chapter 12

FURTHER TECHNIQUES OF DIFFERENTIATION

12.1. The Chain Rule

We begin by re-examining a few examples discussed in the previous chapter.

EXAMPLE 12.1.1. Recall Examples 11.2.5 and 11.2.10, that for the function

$$y = h(x) = (x^2 + 2x)^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 4x^3 + 12x^2 + 8x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = u^2 \qquad \text{and} \qquad u = f(x) = x^2 + 2x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u(2x+2) = 2(x^2+2x)(2x+2) = 4x^3 + 12x^2 + 8x.$$

† This chapter was written at Macquarie University in 1999.

Example 12.1.2. Recall Example 11.3.3, that for the function

$$y = h(x) = \sin^2 x,$$

we have

$$\frac{dy}{dx} = h'(x) = 2\sin x \cos x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = u^2 \qquad \text{and} \qquad u = f(x) = \sin x.$$

Note that

$$\frac{dy}{du} = 2u \qquad \text{and} \qquad \frac{du}{dx} = \cos x,$$

so that

$$\frac{dy}{du} \times \frac{du}{dx} = 2u\cos x = 2\sin x \cos x.$$

Example 12.1.3. Recall Example 11.3.4, that for the function

$$y = h(x) = \sin 2x,$$

we have

$$\frac{dy}{dx} = h'(x) = 2\cos 2x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = 2x.$$

Note that

$$\frac{dy}{du} = \cos u \qquad \text{and} \qquad \frac{du}{dx} = 2,$$

so that

$$\frac{dy}{du} \times \frac{du}{dx} = 2\cos u = 2\cos 2x.$$

In these three examples, we consider functions of the form $y = h(x)$ which can be described in a chain by $y = g(u)$ and $u = f(x)$, where u is some intermediate variable. Suppose that $x_0, x_1 \in \mathbb{R}$. Write $u_0 = f(x_0)$ and $u_1 = f(x_1)$, and write $y_0 = g(u_0)$ and $y_1 = g(u_1)$. Then clearly $h(x_0) = g(f(x_0))$ and $h(x_1) = g(f(x_1))$. Heuristically, we have

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{u_1 - u_0} \times \frac{u_1 - u_0}{x_1 - x_0} = \frac{g(u_1) - g(u_0)}{u_1 - u_0} \times \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

If x_1 is close to x_0, then we expect that u_1 is close to u_0, and so the product

$$\frac{g(u_1) - g(u_0)}{u_1 - u_0} \times \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is close to $g'(u_0)f'(x_0)$, while the product

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0}$$

is close to $h'(x_0)$. It is therefore not unreasonable to expect the following result, although a formal proof is somewhat more complicated.

CHAIN RULE. *Suppose that $y = g(u)$ and $u = f(x)$. Then*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},$$

provided that the two derivatives on the right hand side exist.

We can interpret the rule in the following way. As we vary x, the value $u = f(x)$ changes at the rate of $\mathrm{d}u/\mathrm{d}x$. This change in the value of $u = f(x)$ in turn causes a change in the value of $y = g(u)$ at the rate of $\mathrm{d}y/\mathrm{d}u$.

EXAMPLE 12.1.4. Consider the function $y = h(x) = (x^2 - 6x + 5)^3$. We can set up a chain by writing

$$y = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = x^2 - 6x + 5.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 6,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2(2x-6) = 6(x^2 - 6x + 5)^2(x-3).$$

EXAMPLE 12.1.5. Consider the function $y = h(x) = \sin^4 x$. We can set up a chain by writing

$$y = g(u) = u^4 \qquad \text{and} \qquad u = f(x) = \sin x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 4u^3 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3 \cos x = 4\sin^3 x \cos x.$$

EXAMPLE 12.1.6. Consider the function $y = h(x) = \sec(x^4)$. We can set up a chain by writing

$$y = g(u) = \sec u \qquad \text{and} \qquad u = f(x) = x^4.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \tan u \sec u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 \tan u \sec u = 4x^3 \tan(x^4)\sec(x^4).$$

EXAMPLE 12.1.7. Consider the function $y = h(x) = \tan(x^2 - 3x + 4)$. We can set up a chain by writing

$$y = g(u) = \tan u \qquad \text{and} \qquad u = f(x) = x^2 - 3x + 4.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \sec^2 u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=(2x-3)\sec^2 u=(2x-3)\sec^2(x^2-3x+4).$$

EXAMPLE 12.1.8. Consider the function $y=h(x)=(x^2+5x-1)^{2/3}$. We can set up a chain by writing

$$y=g(u)=u^{2/3}\qquad\text{and}\qquad u=f(x)=x^2+5x-1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{2}{3}u^{-1/3}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=2x+5,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2}{3}u^{-1/3}(2x+5)=\frac{2}{3}(x^2+5x-1)^{-1/3}(2x+5).$$

EXAMPLE 12.1.9. Consider the function

$$y=h(x)=\frac{1}{\cos^3 x}.$$

We can set up a chain by writing

$$y=g(u)=\frac{1}{u^3}=u^{-3}\qquad\text{and}\qquad u=f(x)=\cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-3u^{-4}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=3u^{-4}\sin x=\frac{3\sin x}{u^4}=\frac{3\sin x}{\cos^4 x}.$$

EXAMPLE 12.1.10. Consider the function

$$y=h(x)=\frac{1}{(2x^3-5x+1)^4}.$$

We can set up a chain by writing

$$y=g(u)=\frac{1}{u^4}=u^{-4}\qquad\text{and}\qquad u=f(x)=2x^3-5x+1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-4u^{-5}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=6x^2-5,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=4u^{-5}(5-6x^2)=\frac{4(5-6x^2)}{u^5}=\frac{4(5-6x^2)}{(2x^3-5x+1)^5}.$$

EXAMPLE 12.1.11. Consider the function $y = h(x) = \sin(\cos x)$. We can set up a chain by writing

$$y = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = \cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\cos u \sin x = -\cos(\cos x)\sin x.$$

EXAMPLE 12.1.12. Consider the function

$$y = h(x) = \frac{1}{2(x+1)} + \frac{1}{4(x+1)^2}.$$

We can set up a chain by writing

$$y = g(u) = \frac{1}{2u} + \frac{1}{4u^2} = \frac{1}{2}u^{-1} + \frac{1}{4}u^{-2} \qquad \text{and} \qquad u = f(x) = x + 1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3} = -\frac{1}{2(x+1)^2} - \frac{1}{2(x+1)^3}.$$

EXAMPLE 12.1.13. Consider the function

$$y = h(x) = \left(\frac{x-1}{x+1}\right)^3.$$

We can set up a chain by writing

$$y = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = \frac{x-1}{x+1}.$$

Clearly we have (using the quotient rule for the latter)

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{(x+1)^2},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2 \times \frac{2}{(x+1)^2} = 3\left(\frac{x-1}{x+1}\right)^2 \times \frac{2}{(x+1)^2} = \frac{6(x-1)^2}{(x+1)^4}.$$

The chain rule can be extended to chains of more than two functions. We illustrate the ideas by considering the next four examples.

EXAMPLE 12.1.14. Consider the function $y = h(x) = \sin^3(x^2 + 2)$. We can set up a chain by writing

$$y = k(v) = v^3, \qquad v = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = x^2 + 2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 3v^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 6xv^2 \cos u = 6x \sin^2 u \cos u = 6x \sin^2(x^2+2)\cos(x^2+2).$$

EXAMPLE 12.1.15. Consider the function $y = h(x) = (1 + (1 + x)^{1/2})^5$. We can set up a chain by writing

$$y = k(v) = v^5, \qquad v = g(u) = 1 + u^{1/2} \qquad \text{and} \qquad u = f(x) = 1 + x.$$

Clearly we have

$$\frac{dy}{dv} = 5v^4, \qquad \frac{dv}{du} = \frac{1}{2}u^{-1/2} \qquad \text{and} \qquad \frac{du}{dx} = 1,$$

so it follows from the chain rule that

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx} = \frac{5}{2}v^4 u^{-1/2} = \frac{5}{2}(1 + u^{1/2})^4 u^{-1/2} = \frac{5(1 + (1 + x)^{1/2})^4}{2(1 + x)^{1/2}}.$$

EXAMPLE 12.1.16. Consider the function $y = h(x) = \tan((x^4 - 3x)^3)$. We can set up a chain by writing

$$y = k(v) = \tan v, \qquad v = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = x^4 - 3x.$$

Clearly we have

$$\frac{dy}{dv} = \sec^2 v, \qquad \frac{dv}{du} = 3u^2 \qquad \text{and} \qquad \frac{du}{dx} = 4x^3 - 3,$$

so it follows from the chain rule that

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx} &= 3u^2(4x^3 - 3)\sec^2 v = 3u^2(4x^3 - 3)\sec^2(u^3) \\ &= 3(x^4 - 3x)^2(4x^3 - 3)\sec^2((x^4 - 3x)^3).\end{aligned}$$

EXAMPLE 12.1.17. Consider the function $y = h(x) = \sqrt{x^2 + \sin(x^2)}$. We can set up a chain by writing

$$y = k(v) = v^{1/2}, \qquad v = g(u) = u + \sin u \qquad \text{and} \qquad u = f(x) = x^2.$$

Clearly we have

$$\frac{dy}{dv} = \frac{1}{2v^{1/2}}, \qquad \frac{dv}{du} = 1 + \cos u \qquad \text{and} \qquad \frac{du}{dx} = 2x,$$

so it follows from the chain rule that

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx} = \frac{x(1 + \cos u)}{v^{1/2}} = \frac{x(1 + \cos u)}{(u + \sin u)^{1/2}} = \frac{x(1 + \cos(x^2))}{\sqrt{x^2 + \sin(x^2)}}.$$

We conclude this section by studying three examples where the chain rule is used only in part of the argument. These examples are rather hard, and the reader is advised to concentrate on the ideas and not to get overly worried about the arithmetic details. For accuracy, it is absolutely crucial that we exercise great care.

EXAMPLE 12.1.18. Consider the function $y = h(x) = (x^2 - 1)^{1/2}(x^2 + 4x + 3)$. We can write

$$h(x) = f(x)g(x),$$

where

$$f(x) = (x^2 - 1)^{1/2} \qquad \text{and} \qquad g(x) = x^2 + 4x + 3.$$

It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

It is easy to see that $g'(x) = 2x + 4$. To find $f'(x)$, we shall use the chain rule. Let

$$z = f(x) = (x^2 - 1)^{1/2}.$$

We can set up a chain by writing

$$z = u^{1/2} \qquad \text{and} \qquad u = x^2 - 1.$$

Clearly we have

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{1}{2u^{1/2}} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$f'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{u^{1/2}} = \frac{x}{(x^2 - 1)^{1/2}}.$$

Hence

$$h'(x) = \frac{x(x^2 + 4x + 3)}{(x^2 - 1)^{1/2}} + (x^2 - 1)^{1/2}(2x + 4).$$

EXAMPLE 12.1.19. Consider the function

$$y = h(x) = \frac{(1 - x^3)^2}{(1 + 2x + 3x^2)^2}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where

$$f(x) = (1 - x^3)^2 \qquad \text{and} \qquad g(x) = (1 + 2x + 3x^2)^2.$$

It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

To find $f'(x)$ and $g'(x)$, we shall use the chain rule. Let

$$z = f(x) = (1 - x^3)^2 \qquad \text{and} \qquad w = g(x) = (1 + 2x + 3x^2)^2.$$

We can set up a chain by writing

$$z = u^2 \qquad \text{and} \qquad u = 1 - x^3.$$

Then

$$f'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u \times (-3x^2) = -6ux^2 = -6(1 - x^3)x^2.$$

Similarly, we can set up a chain by writing

$$w = v^2 \qquad \text{and} \qquad v = 1 + 2x + 3x^2.$$

Then

$$g'(x) = \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 2v \times (2 + 6x) = 4v(1 + 3x) = 4(1 + 2x + 3x^2)(1 + 3x).$$

Hence

$$h'(x) = \frac{-6(1+2x+3x^2)^2(1-x^3)x^2 - 4(1-x^3)^2(1+2x+3x^2)(1+3x)}{(1+2x+3x^2)^4}$$
$$= -\frac{6(1+2x+3x^2)(1-x^3)x^2 + 4(1-x^3)^2(1+3x)}{(1+2x+3x^2)^3}.$$

Alternatively, observe that we can set up a chain by writing

$$y = s^2 \qquad \text{and} \qquad s = \frac{1-x^3}{1+2x+3x^2}.$$

Clearly we have (using the quotient rule for the latter)

$$\frac{\mathrm{d}y}{\mathrm{d}s} = 2s \qquad \text{and} \qquad \frac{\mathrm{d}s}{\mathrm{d}x} = \frac{-3(1+2x+3x^2)x^2 - (1-x^3)(2+6x)}{(1+2x+3x^2)^2},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}x} = -2s \times \frac{3(1+2x+3x^2)x^2 + (1-x^3)(2+6x)}{(1+2x+3x^2)^2}$$
$$= -\frac{2(1-x^3)}{1+2x+3x^2} \times \frac{3(1+2x+3x^2)x^2 + 2(1-x^3)(1+3x)}{(1+2x+3x^2)^2}.$$

It can be easily checked that the answer is the same as before.

EXAMPLE 12.1.20. Consider the function $y = h(x) = (x^2 + (x^3 + x^5)^7)^{11}$. We can set up a chain by writing

$$y = g(u) = u^{11} \qquad \text{and} \qquad u = f(x) = x^2 + (x^3 + x^5)^7.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 11u^{10}.$$

On the other hand, we have $f(x) = k(x) + t(x)$, where $k(x) = x^2$ and $t(x) = (x^3 + x^5)^7$. It follows that $f'(x) = k'(x) + t'(x)$. Note that $k'(x) = 2x$. To find $t'(x)$, we shall use the chain rule. Let

$$z = t(x) = (x^3 + x^5)^7.$$

We can set up a chain by writing

$$z = v^7 \qquad \text{and} \qquad v = x^3 + x^5.$$

Clearly we have

$$\frac{\mathrm{d}z}{\mathrm{d}v} = 7v^6 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 3x^2 + 5x^4,$$

so it follows from the chain rule that

$$t'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 7v^6(3x^2 + 5x^4) = 7(x^3 + x^5)^6(3x^2 + 5x^4),$$

and so

$$f'(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 7(x^3 + x^5)^6(3x^2 + 5x^4).$$

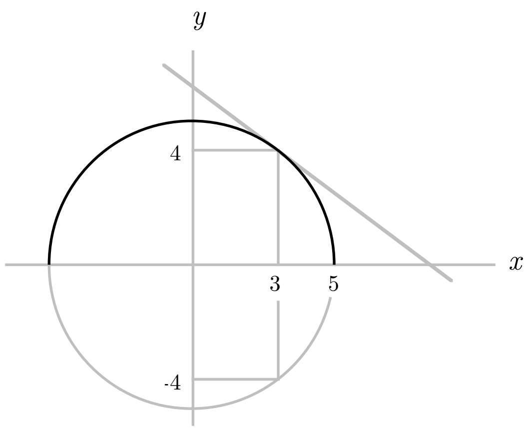
It then follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 11u^{10}(2x + 7(x^3 + x^5)^6(3x^2 + 5x^4))$$
$$= 11(x^2 + (x^3 + x^5)^7)^{10}(2x + 7(x^3 + x^5)^6(3x^2 + 5x^4)).$$

12.2. Implicit Differentiation

A function $y = f(x)$ can usually be viewed as a curve on the xy-plane, and gives a relationship between the (independent) variable x and the (dependent) variable y by describing y explicitly in terms of x. However, a relationship between two variables x and y cannot always be expressed as a function $y = f(x)$. Moreover, we may even choose to describe a function $y = f(x)$ implicitly by simply giving some relationship between the variables x and y, and not describing y explicitly in terms of x.

EXAMPLE 12.2.1. Consider the equation $x^2 + y^2 = 25$, representing a circle of radius 5 and centred at the origin $(0, 0)$. This equation expresses a relationship between the two variables x and y, but y is not given explicitly in terms of x. Indeed, it is not possible to give y explicitly in terms of x, as this equation does not represent a function $y = f(x)$. To see this, note that if $x = 3$, then both $y = 4$ and $y = -4$ will satisfy the equation, so it is meaningless to talk of $f(3)$. On the other hand, we see that the point $(3, 4)$ is on the circle, and clearly there is a tangent line to the circle at the point $(3, 4)$, as shown in the picture below.



If we restrict our attention to the upper semicircle, then we can express the variable y explicitly as a function of the variable x by writing

$$y = (25 - x^2)^{1/2}.$$

We can set up a chain by writing

$$y = u^{1/2} \qquad \text{and} \qquad u = 25 - x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2}u^{-1/2} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -xu^{-1/2} = -x(25 - x^2)^{-1/2}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

Note that our argument here involves obtaining an explicit expression for the variable y in terms of the variable x from similar information given implicitly by the equation $x^2+y^2 = 25$. Now let us see whether we can obtain a similar conclusion concerning the slope of the tangent line at the point $(3, 4)$ without first having to obtain the explicit expression $y = (25 - x^2)^{1/2}$ of the upper semicircle. Let us start from the equation $x^2 + y^2 = 25$ of the circle. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(25).$$

Using the rule on the derivatives of constants, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(25) = 0.$$

Using the sum rule and the rule on the derivatives of powers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2x + \frac{\mathrm{d}}{\mathrm{d}x}(y^2).$$

We next set up a chain by writing

$$z = y^2 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then using the chain rule and the rule on the derivatives of powers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(3,4)} = -\frac{3}{4}$$

as before.

The second part of the example above is a case of using implicit differentiation, where we find the derivative of a function $y = f(x)$ without knowing any explicit expression for the variable y in terms of the variable x. We shall describe this technique further by discussing a few more examples. In some of these examples, it may be very difficult, if not impossible, to find any explicit expression for the variable y in terms of the variable x.

EXAMPLE 12.2.2. Suppose that $y^2 - x^2 = 4$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2 - x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2 - x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2) - \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 2x = 0.$$

We next set up a chain by writing

$$z = y^2 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}.$$

EXAMPLE 12.2.3. Suppose that $y^3 + \sin x = 3$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3 + \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3 + \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(y^3) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(y^3) + \cos x = 0.$$

We next set up a chain by writing

$$z = y^3 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + \cos x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos x}{3y^2}.$$

EXAMPLE 12.2.4. Suppose that $y^5 + 3y^2 - 2x^2 + 4 = 0$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5 + 3y^2 - 2x^2 + 4) = \frac{\mathrm{d}}{\mathrm{d}x}(0) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5 + 3y^2 - 2x^2 + 4) = \frac{\mathrm{d}}{\mathrm{d}x}(y^5) + 3\frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 2\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(4) = \frac{\mathrm{d}}{\mathrm{d}x}(y^5) + 3\frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 4x = 0.$$

Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5) = \frac{\mathrm{d}}{\mathrm{d}y}(y^5) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$(5y^4 + 6y)\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{5y^4 + 6y}.$$

EXAMPLE 12.2.5. Suppose that $xy = 6$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \frac{\mathrm{d}}{\mathrm{d}x}(6) = 0.$$

It follows from the product rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times y + x \times \frac{\mathrm{d}}{\mathrm{d}x}(y) = y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}.$$

EXAMPLE 12.2.6. Suppose that $x^3+2x^2y^3+3y^4=6$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x^2y^3+3y^4)=\frac{\mathrm{d}}{\mathrm{d}x}(6)=0.$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x^2y^3+3y^4)&=\frac{\mathrm{d}}{\mathrm{d}x}(x^3)+2\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\times y^3+x^2\times\frac{\mathrm{d}}{\mathrm{d}x}(y^3)\right)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^4)\\&=3x^2+4xy^3+2x^2\frac{\mathrm{d}}{\mathrm{d}x}(y^3)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^4)=0.\end{aligned}$$

Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=\frac{\mathrm{d}}{\mathrm{d}y}(y^3)\times\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}x}(y^4)=\frac{\mathrm{d}}{\mathrm{d}y}(y^4)\times\frac{\mathrm{d}y}{\mathrm{d}x}=4y^3\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$3x^2+4xy^3+(6x^2y^2+12y^3)\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3x^2+4xy^3}{6x^2y^2+12y^3}.$$

Note next that the point $(1,1)$ satisfies the equation. It follows that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)}=-\frac{7}{18}.$$

Check that the equation of the tangent line at this point is given by $7x+18y=25$.

EXAMPLE 12.2.7. Suppose that $(x^2+y^3)^2=9$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}((x^2+y^3)^2)=\frac{\mathrm{d}}{\mathrm{d}x}(9)=0.$$

Let $w=(x^2+y^3)^2$. We can set up a chain by writing

$$w=z^2\qquad\text{and}\qquad z=x^2+y^3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}((x^2+y^3)^2)=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{\mathrm{d}w}{\mathrm{d}z}\times\frac{\mathrm{d}z}{\mathrm{d}x}=2z\frac{\mathrm{d}z}{\mathrm{d}x}=2(x^2+y^3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3).$$

Hence

$$(x^2+y^3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3)=0.$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3)=\frac{\mathrm{d}}{\mathrm{d}x}(x^2)+\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x},$$

where we have used the chain rule at the last step. Summarizing, we obtain

$$(x^2+y^3)\left(2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0.$$

It is clear that $x^2+y^3\neq0$ for any point (x,y) satisfying the equation. It follows that

$$2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{3y^2}.$$

Note next that the point $(2, -1)$ satisfies the equation. It follows that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(2,-1)} = -\frac{4}{3}.$$

Check that the equation of the tangent line at this point is given by $4x + 3y = 5$.

EXAMPLE 12.2.8. The point $(1, 1)$ is one of the intersection points of the parabola $y - x^2 = 0$ and the ellipse $x^2 + 2y^2 = 3$. We shall show that the two tangents at $(1, 1)$ are perpendicular to each other. Consider first of all the parabola $y - x^2 = 0$. Here we can write $y = x^2$, so that $\mathrm{d}y/\mathrm{d}x = 2x$. Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = 2.$$

Consider next the ellipse $x^2 + 2y^2 = 3$. Using implicit differentiation, it is not difficult to show that

$$2x + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{2y}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = -\frac{1}{2}.$$

Since the product of the two derivatives is equal to -1, it follows that the two tangents are perpendicular to each other.

12.3. Derivatives of the Exponential and Logarithmic Functions

We shall state without proof the following result.

DERIVATIVE OF THE EXPONENTIAL FUNCTION. *If $f(x) = \mathrm{e}^x$, then $f'(x) = \mathrm{e}^x$.*

EXAMPLE 12.3.1. Consider the function $y = h(x) = \mathrm{e}^x(\sin x + 2\cos x)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = \mathrm{e}^x$ and $g(x) = \sin x + 2\cos x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly

$$f'(x) = \mathrm{e}^x \qquad \text{and} \qquad g'(x) = \cos x - 2\sin x.$$

Hence

$$h'(x) = \mathrm{e}^x(\sin x + 2\cos x) + \mathrm{e}^x(\cos x - 2\sin x) = \mathrm{e}^x(3\cos x - \sin x).$$

EXAMPLE 12.3.2. Consider the function $y = h(x) = \mathrm{e}^x(x^2 + x + 2)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = \mathrm{e}^x$ and $g(x) = x^2 + x + 2$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly

$$f'(x) = \mathrm{e}^x \qquad \text{and} \qquad g'(x) = 2x + 1.$$

Hence

$$h'(x) = \mathrm{e}^x(x^2 + x + 2) + \mathrm{e}^x(2x + 1) = \mathrm{e}^x(x^2 + 3x + 3).$$

EXAMPLE 12.3.3. Consider the function $y = h(x) = \mathrm{e}^{2x}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = 2x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^u = 2\mathrm{e}^{2x}.$$

Alternatively, we can set up a chain by writing

$$y = t(v) = v^2 \qquad \text{and} \qquad v = k(x) = \mathrm{e}^x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 2v \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 2v\mathrm{e}^x = 2\mathrm{e}^x\mathrm{e}^x = 2\mathrm{e}^{2x}.$$

Yet another alternative is to observe that $h(x) = \mathrm{e}^x\mathrm{e}^x$. It follows that we can use the product rule instead of the chain rule. Try it!

EXAMPLE 12.3.4. Consider the function $y = h(x) = \mathrm{e}^{x^3}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = x^3.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2\mathrm{e}^u = 3x^2\mathrm{e}^{x^3}.$$

EXAMPLE 12.3.5. Consider the function $y = h(x) = \mathrm{e}^{\sin x + 4\cos x}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = \sin x + 4\cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x - 4\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u(\cos x - 4\sin x) = \mathrm{e}^{\sin x + 4\cos x}(\cos x - 4\sin x).$$

EXAMPLE 12.3.6. Consider the function

$$y = h(x) = \sin^3\left(\mathrm{e}^{4x^2}\right).$$

We can set up a chain by writing

$$y = t(w) = w^3, \qquad w = k(v) = \sin v, \qquad v = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = 4x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}w} = 3w^2, \qquad \frac{\mathrm{d}w}{\mathrm{d}v} = \cos v, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 8x,$$

so it follows from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}w} \times \frac{\mathrm{d}w}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} &= 24x\mathrm{e}^u w^2 \cos v = 24x\mathrm{e}^u \sin^2 v \cos v = 24x\mathrm{e}^u \sin^2(\mathrm{e}^u)\cos(\mathrm{e}^u)\\ &= 24x\mathrm{e}^{4x^2}\sin^2(\mathrm{e}^{4x^2})\cos(\mathrm{e}^{4x^2}).\end{aligned}$$

EXAMPLE 12.3.7. Suppose that $\mathrm{e}^{2x} + y^2 = 5$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(5) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2\mathrm{e}^{2x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

using Example 12.3.3 and the chain rule. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\mathrm{e}^{2x}}{y}.$$

The next example is rather complicated, and the reader is advised to concentrate on the ideas and not to get overly worried about the arithmetic details. For accuracy, it is absolutely crucial that we exercise great care.

EXAMPLE 12.3.8. Suppose that $\mathrm{e}^{2x}\sin 3y + x^2y^3 = 3$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}\sin 3y + x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0.$$

Using the sum and product rules, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}\sin 3y + x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) \times \sin 3y + \mathrm{e}^{2x} \times \frac{\mathrm{d}}{\mathrm{d}x}(\sin 3y) + \frac{\mathrm{d}}{\mathrm{d}x}(x^2) \times y^3 + x^2 \times \frac{\mathrm{d}}{\mathrm{d}x}(y^3).$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) = 2\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x.$$

Writing $z = 3y$ and using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3y) = \frac{\mathrm{d}}{\mathrm{d}y}(\sin 3y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}(\sin z) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}z}(\sin z) \times \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos z\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos 3y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Using the chain rule, we also obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = \frac{\mathrm{d}}{\mathrm{d}y}(y^3) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$2\mathrm{e}^{2x}\sin 3y + 3\mathrm{e}^{2x}\cos 3y\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^3 + 3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\mathrm{e}^{2x}\sin 3y + xy^3) + 3(\mathrm{e}^{2x}\cos 3y + x^2y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2(\mathrm{e}^{2x}\sin 3y + xy^3)}{3(\mathrm{e}^{2x}\cos 3y + x^2y^2)}.$$

Next, we turn to the logarithmic function. Using implicit differentiation, we can establish the following result.

DERIVATIVE OF THE LOGARITHMIC FUNCTION. *If $f(x) = \log x$, then $f'(x) = 1/x$.*

PROOF. Suppose that $y = \log x$. Then $\mathrm{e}^y = x$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1.$$

Using the chain rule and the rule on the derivative of the exponential function, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^y} = \frac{1}{x}. \qquad ♣$$

EXAMPLE 12.3.9. Consider the function $y = h(x) = x\log x$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x$ and $g(x) = \log x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly $f'(x) = 1$ and $g'(x) = 1/x$. Hence $h'(x) = \log x + 1$.

Example 12.3.10. Consider the function

$$y = h(x) = \frac{x \log x + \sin x}{\mathrm{e}^x}.$$

We can write

$$h(x) = \frac{f(x) + k(x)}{g(x)},$$

where $f(x) = x \log x$, $k(x) = \sin x$ and $g(x) = \mathrm{e}^x$. It follows from the sum and quotient rules that

$$h'(x) = \frac{g(x)(f'(x) + k'(x)) - (f(x) + k(x))g'(x)}{g^2(x)}.$$

Clearly $k'(x) = \cos x$ and $g'(x) = \mathrm{e}^x$. Observe also from Example 12.3.9 that $f'(x) = \log x + 1$. Hence

$$h'(x) = \frac{\mathrm{e}^x(\log x + 1 + \cos x) - (x \log x + \sin x)\mathrm{e}^x}{\mathrm{e}^{2x}} = \frac{(1 - x) \log x + 1 + \cos x - \sin x}{\mathrm{e}^x}.$$

Example 12.3.11. Consider the function $y = h(x) = \log(5x^2 + 3)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = 5x^2 + 3.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 10x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{10x}{u} = \frac{10x}{5x^2 + 3}.$$

Example 12.3.12. Consider the function $y = h(x) = \log(\tan x + \sec x)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = \tan x + \sec x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x + \tan x \sec x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sec^2 x + \tan x \sec x}{u} = \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} = \sec x.$$

Example 12.3.13. Consider the function $y = h(x) = \log(\cot x + \csc x)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = \cot x + \csc x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\csc^2 x - \cot x \csc x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-\csc^2 x - \cot x \csc x}{u} = -\frac{\csc^2 x + \cot x \csc x}{\cot x + \csc x} = -\csc x.$$

EXAMPLE 12.3.14. Consider the function $y = h(x) = \log(\sin(x^{1/2}))$. We can set up a chain by writing

$$y = k(v) = \log v, \qquad v = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = x^{1/2}.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{1}{v}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2x^{1/2}},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cos u}{2x^{1/2}v} = \frac{\cos u}{2x^{1/2}\sin u} = \frac{\cos(x^{1/2})}{2x^{1/2}\sin(x^{1/2})}.$$

EXAMPLE 12.3.15. Suppose that $x \log y + y^2 = 4$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \log y + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \log y + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times \log y + x \times \frac{\mathrm{d}}{\mathrm{d}x}(\log y) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \log y + x\frac{\mathrm{d}}{\mathrm{d}x}(\log y) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2).$$

By the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log y) = \frac{\mathrm{d}}{\mathrm{d}y}(\log y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$\log y + \left(\frac{x}{y} + 2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y \log y}{x + 2y^2}.$$

EXAMPLE 12.3.16. Suppose that $\log(xy^2) = 2x^2$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(xy^2)) = \frac{\mathrm{d}}{\mathrm{d}x}(2x^2) = 4x.$$

Let $z = \log(xy^2)$. We can set up a chain by writing

$$z = \log u \qquad \text{and} \qquad u = xy^2.$$

Then it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(xy^2)) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{xy^2} \times \frac{\mathrm{d}}{\mathrm{d}x}(xy^2).$$

Next, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times y^2 + x \times \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = y^2 + x \times \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^2y^2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4x^2 - 1)y^2}{2xy}.$$

12.4. Derivatives of the Inverse Trigonometric Functions

The purpose of this last section is to determine the derivatives of the inverse trigonometric functions by using implicit differentiation and our knowledge on the derivatives of the trigonometric functions.

For notational purposes, we shall write

$$y = \sin^{-1} x \qquad \text{if and only if} \qquad x = \sin y,$$

and similarly for the other trigonometric functions. These inverse trigonometric functions are well defined, provided that we restrict the values for x to suitable intervals of real numbers.

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS.

(a) If $y = \sin^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{1-x^2}}$.

(b) If $y = \cos^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{\sqrt{1-x^2}}$.

(c) If $y = \tan^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1+x^2}$.

(d) If $y = \cot^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{1+x^2}$.

(e) If $y = \sec^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x\sqrt{x^2-1}}$.

(f) If $y = \csc^{-1} x$, *then* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{x\sqrt{x^2-1}}$.

SKETCH OF PROOF. For simplicity, we shall assume that $0 < y < \pi/2$, so that y is in the first quadrant, and so all the trigonometric functions have positive values.

(a) If $y = \sin^{-1} x$, then $x = \sin y$. Differentiating with respect to x, we obtain

$$1 = \cos y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

(b) If $y = \cos^{-1} x$, then $x = \cos y$. Differentiating with respect to x, we obtain

$$1 = -\sin y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$

(c) If $y = \tan^{-1} x$, then $x = \tan y$. Differentiating with respect to x, we obtain

$$1 = \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

(d) If $y = \cot^{-1} x$, then $x = \cot y$. Differentiating with respect to x, we obtain

$$1 = -\csc^2 y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}.$$

(e) If $y=\sec^{-1}x$, then $x=\sec y$. Differentiating with respect to x, we obtain

$$1=\tan y\sec y\frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\tan y\sec y}=\frac{1}{(\sec^2 y-1)^{1/2}\sec y}=\frac{1}{x\sqrt{x^2-1}}.$$

(f) If $y=\csc^{-1}x$, then $x=\csc y$. Differentiating with respect to x, we obtain

$$1=-\cot y\csc y\frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\cot y\csc y}=-\frac{1}{(\csc^2 y-1)^{1/2}\csc y}=-\frac{1}{x\sqrt{x^2-1}}.$$ ♣

There is no need to remember the derivatives of any of these inverse trigonometric functions.

Problems for Chapter 12

1. By making suitable use of the chain rule and other rules as appropriate, find the derivative of each of the following functions:
   a) $h(x)=\sqrt{1-\cos x}$ b) $h(x)=\sin(3x)$ c) $h(x)=\cos(\sin x)$
   d) $h(x)=x^2\cos x$ e) $h(x)=\sin(2x)\sin(3x)$ f) $h(x)=2x\sin(3x)$
   g) $h(x)=\tan(3x)$ h) $h(x)=4\sec(5x)$ i) $h(x)=\cos(x^3)$
   j) $h(x)=\cos^3 x$ k) $h(x)=(1+\cos^2 x)^6$ l) $h(x)=\tan(x^2)+\tan^2 x$
   m) $h(x)=\cos(\tan x)$ n) $h(x)=\sin(\sin x)$

2. Find the derivative of each of the following functions:
   a) $h(x)=\sin(\mathrm{e}^x)$ b) $h(x)=\mathrm{e}^{\sin x}$ c) $h(x)=\mathrm{e}^{-2x}\sin x$
   d) $h(x)=\mathrm{e}^x\sin(2x)$ e) $h(x)=\tan(\mathrm{e}^{-3x})$ f) $h(x)=\tan(\mathrm{e}^x)$

3. a) For each $k=0,1,2,3,\ldots$, find a function $f_k(x)$ such that $f_k'(x)=x^k$.
   b) For each $k=-2,-3,-4,\ldots$, find a function $f_k(x)$ such that $f_k'(x)=x^k$.
   c) Find a function $f_{-1}(x)$ such that $f_{-1}'(x)=x^{-1}$.

4. Find the derivative of each of the following functions:
   a) $h(x)=(3x^2+\pi)(\mathrm{e}^x-4)$ b) $h(x)=x^5+3x^2+\dfrac{2}{x^4}+1$ c) $h(x)=2x-\dfrac{1}{\sqrt[3]{x}}+\mathrm{e}^{2x}$
   d) $h(x)=2\mathrm{e}^x+x\mathrm{e}^{3x}$ e) $h(x)=\mathrm{e}^{\tan x}$ f) $h(x)=2x\mathrm{e}^x-x^{-2}$
   g) $h(x)=\log(\log(2x^3))$ h) $h(x)=\sqrt{x+5}$ i) $h(x)=\dfrac{x+2}{x^2+1}$
   j) $h(x)=\sin(2x+3)$ k) $h(x)=\cos^2(2x)$ l) $h(x)=\log(\mathrm{e}^{-x}-1)$
   m) $h(x)=\mathrm{e}^{\mathrm{e}^x+\mathrm{e}^{-x}}$ n) $h(x)=\dfrac{x^2+1}{\sqrt{x}}$ o) $h(x)=(x+3)^2$
   p) $h(x)=\log(2x+3)$ q) $h(x)=\tan(3x+2x^2)$ r) $h(x)=\cos(\mathrm{e}^{2x})$

5. Use implicit differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following relations:
   a) $x^2+xy-y^3=xy^2$ b) $x^2+y^2=\sqrt{7}$ c) $\sqrt{x}+\sqrt{y}=25$
   d) $\sin(xy)=2x+5$ e) $x\log y+y^3=\log x$ f) $y^3-xy=-6$
   g) $x^2-xy+y^4=x^2y$ h) $\sin(xy)=3x^2-2$

6. For each of the following, verify first that the given point satisfies the relation defining the curve, then find the equation of the tangent line to the curve at the point:

   a) $xy^2 = 1$ at $(1, 1)$

   b) $y^2 = \dfrac{x^2}{xy - 4}$ at $(4, 2)$

   c) $y + \sin y + x^2 = 9$ at $(3, 0)$

   d) $x^{2/3} + y^{2/3} = a^{2/3}$ at $(a, 0)$

– ∗ – ∗ – ∗ – ∗ – ∗ –