ELEMENTARY MATHEMATICS

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Chapter 13

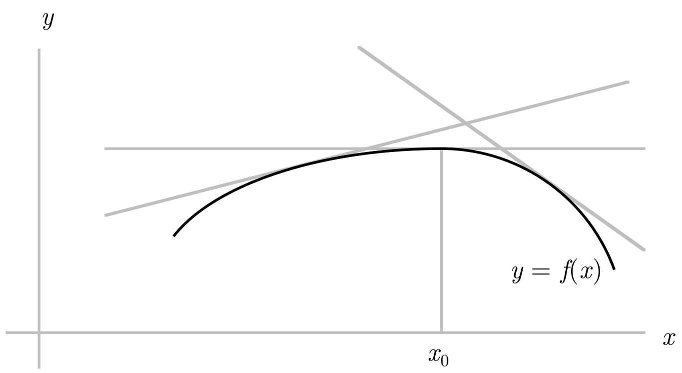
APPLICATIONS OF DIFFERENTIATION

13.1. Second Derivatives

Recall that for a function $y = f(x)$, the derivative $f'(x)$ represents the slope of the tangent. It is easy to see from a picture that if the derivative $f'(x) > 0$, then the function $f(x)$ is increasing; in other words, $f(x)$ increases in value as x increases. On the other hand, if the derivative $f'(x) < 0$, then the function $f(x)$ is decreasing; in other words, $f(x)$ decreases in value as x increases. We are interested in the case when the derivative $f'(x) = 0$. Values $x = x_0$ such that $f'(x_0) = 0$ are called stationary points.

Let us introduce the second derivative $f''(x)$ of the function $f(x)$. This is defined to be the derivative of the derivative $f'(x)$. With the same reasoning as before but applied to the function $f'(x)$ instead of the function $f(x)$, we conclude that if the second derivative $f''(x) > 0$, then the derivative $f'(x)$ is increasing. Similarly, if the second derivative $f''(x) < 0$, then the derivative $f'(x)$ is decreasing.

Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. The condition $f''(x_0) < 0$ tells us that the derivative $f'(x)$ is decreasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f'(x) > 0$ when x is a little smaller than x_0, and that $f'(x) < 0$ when x is a little greater than x_0, as indicated in the picture below.

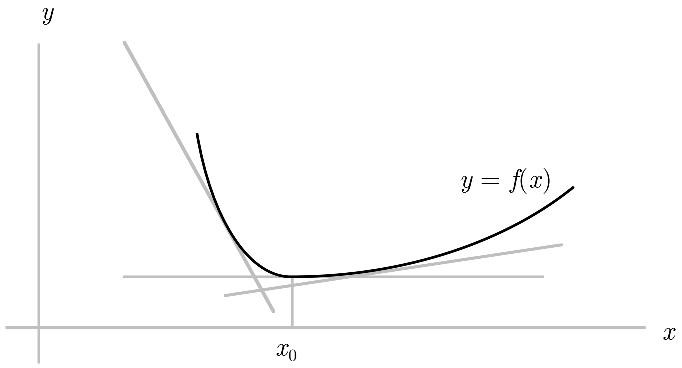


† This chapter was written at Macquarie University in 1999.

In this case, we say that the function has a local maximum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \le f(x_0)$ for all such real values x.

LOCAL MAXIMUM. *Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. Then the function $f(x)$ has a local maximum at the point $x = x_0$.*

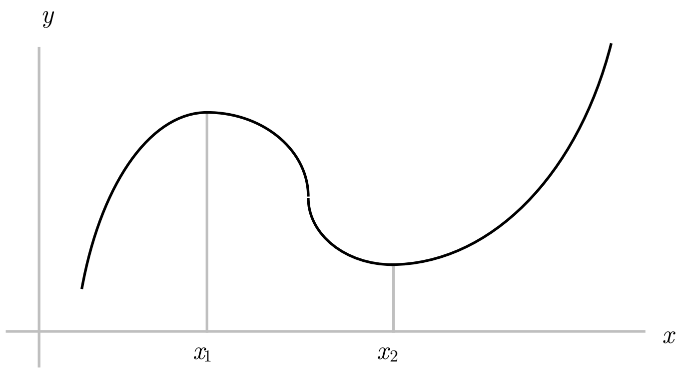
Suppose next that $f'(x_0) = 0$ and $f''(x_0) > 0$. The condition $f''(x_0) > 0$ tells us that the derivative $f'(x)$ is increasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f'(x) < 0$ when x is a little smaller than x_0, and that $f'(x) > 0$ when x is a little greater than x_0, as indicated in the picture below.



In this case, we say that the function has a local minimum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \ge f(x_0)$ for all such real values x.

LOCAL MINIMUM. *Suppose that $f'(x_0) = 0$ and $f''(x_0) > 0$. Then the function $f(x)$ has a local minimum at the point $x = x_0$.*

REMARK. These stationary points are called local maxima or local minima because such points may not maximize or minimize the functions in question. Consider the picture below, with a local maximum at $x = x_1$ and a local minumum at $x = x_2$.



We also say that a point $x = x_0$ is a point of inflection if $f''(x_0) = 0$, irrespective of whether $f'(x_0) = 0$ or not. A simple way of visualizing the graph of a function at a point of inflection is to imagine that one is steering a car along the curve. A point of inflection then corresponds to the place on the curve where the steering wheel of the car is momentarily straight while being turned from a little left to a little right, or while being turned from a little right to a little left.

EXAMPLE 13.1.1. Consider the function $f(x) = \cos x$. Since $f'(x) = -\sin x = 0$ whenever $x = k\pi$, where $k \in \mathbb{Z}$, it follows that the function $f(x) = \cos x$ has a stationary point at $x = k\pi$ for every $k \in \mathbb{Z}$. Next, note that $f''(x) = -\cos x$. If k is even, then $f''(k\pi) = -1$, so that $f(x)$ has a local maximum at $x = k\pi$. If k is odd, then $f''(k\pi) = 1$, so that $f(x)$ has a local minimum at $x = k\pi$. See the graph of this function in Chapter 3.

EXAMPLE 13.1.2. Consider the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$. Since

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x - 1)(x + 2),$$

it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -2$. On the other hand, we have $f''(x) = 36x^2 + 24x - 24$. Since $f''(0) = -24$, $f''(1) = 36$ and $f''(-2) = 72$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -2$.

EXAMPLE 13.1.3. Consider the function $f(x) = x^3 - 3x^2 + 2$. Since $f'(x) = 3x^2 - 6x = 3x(x - 2)$, it follows that the function $f(x)$ has stationary points at $x = 0$ and $x = 2$. On the other hand, we have $f''(x) = 6x - 6$. Since $f''(0) = -6$ and $f''(2) = 6$, it follows that $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$. Observe also that there is a point of inflection at $x = 1$.

EXAMPLE 13.1.4. Consider the function $f(x) = x^4 - 2x^2 + 7$. Since

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1),$$

it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -1$. On the other hand, we have $f''(x) = 12x^2 - 4$. Since $f''(0) = -4$, $f''(1) = 8$ and $f''(-1) = 8$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -1$. Note also that $f''(x) = 0$ if $x = \pm\sqrt{1/3}$, so there are points of inflection at $x = \pm\sqrt{1/3}$.

EXAMPLE 13.1.5. Consider the function $f(x) = 3x^4 - 16x^3 + 24x^2 - 1$. Since

$$f'(x) = 12x^3 - 48x^2 + 48x = 12x(x^2 - 4x + 4) = 12x(x - 2)^2,$$

it follows that the function $f(x)$ has stationary points at $x = 0$ and $x = 2$. On the other hand, we have $f''(x) = 36x^2 - 96x + 48$. Since $f''(0) = 48$ and $f''(2) = 0$, it follows that $f(x)$ has a local minimum at $x = 0$ and a point of inflection at $x = 2$. Note also that $36x^2 - 96x + 48 = 12(x - 2)(3x - 2)$, so there is another point of inflection at $x = 2/3$.

EXAMPLE 13.1.6. Consider the function

$$f(x) = \frac{1}{x^2 + 1}, \qquad \text{with} \qquad f'(x) = -\frac{2x}{(x^2 + 1)^2}.$$

Clearly $f(x)$ has a stationary point at $x = 0$. On the other hand, it is easy to check that

$$f''(x) = \frac{6x^2 - 2}{(x^2 + 1)^3}.$$

Since $f''(0) = -2$, it follows that $f(x)$ has a local maximum at $x = 0$. We also have points of inflection when $6x^2 - 2 = 0$; in other words, when $x = \pm\sqrt{1/3}$.

EXAMPLE 13.1.7. Consider the function

$$f(x) = \frac{x}{x^2 + 1}, \qquad \text{with} \qquad f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Clearly $f(x)$ has stationary points at $x = 1$ and $x = -1$. On the other hand, it is easy to check that

$$f''(x) = -\frac{2x(x^2 + 1) + 4x(1 - x^2)}{(x^2 + 1)^3} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}.$$

Since $f''(1) = -1/2$ and $f''(-1) = 1/2$, it follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = -1$. We also have points of inflection when $2x(x^2 - 3) = 0$; in other words, when $x = 0$ or $x = \pm\sqrt{3}$.

EXAMPLE 13.1.8. Consider the function $f(x) = \mathrm{e}^x + \mathrm{e}^{-x}$. Since $f'(x) = \mathrm{e}^x - \mathrm{e}^{-x}$, it follows that the function $f(x)$ has a stationary point at $x = 0$. On the other hand, we have $f''(x) = \mathrm{e}^x + \mathrm{e}^{-x}$. Since $f''(0) = 2$, it follows that $f(x)$ has a local minimum at $x = 0$.

EXAMPLE 13.1.9. Consider the function $f(x) = \sin x - \cos^2 x$, restricted to the interval $0 \le x \le 2\pi$. It is easy to see that

$$f'(x) = \cos x + 2\cos x \sin x = (1 + 2\sin x)\cos x.$$

We therefore have stationary points when $\cos x = 0$ or $\sin x = -1/2$. There are four stationary points in the interval $0 \le x \le 2\pi$, namely

$$x = \frac{\pi}{2}, \qquad x = \frac{3\pi}{2}, \qquad x = \frac{7\pi}{6}, \qquad x = \frac{11\pi}{6}.$$

Next, note that we can write $f'(x) = \cos x + \sin 2x$, so that $f''(x) = 2\cos 2x - \sin x$. It is easy to check that

$$f''\left(\frac{\pi}{2}\right) = -3, \qquad f''\left(\frac{3\pi}{2}\right) = -1, \qquad f''\left(\frac{7\pi}{6}\right) = \frac{3}{2}, \qquad f''\left(\frac{11\pi}{6}\right) = \frac{3}{2}.$$

Hence $f(x)$ has local maxima at $x = \pi/2$ and $x = 3\pi/2$, and local minima at $x = 7\pi/6$ and $x = 11\pi/6$.

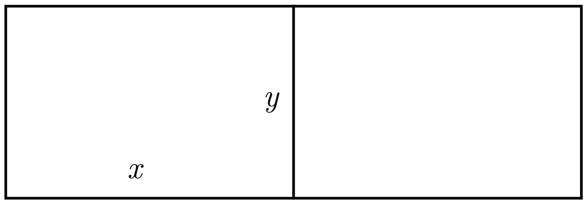
13.2. Applications to Problem Solving

In this section, we discuss how we can apply ideas in differentiation to solve various problems. We shall illustrate the techniques by discussing a few examples. Central to all of these is the crucial step where we set up the problems mathematically and in a suitable way.

EXAMPLE 13.2.1. We wish to find positive real numbers x and y such that $x + y = 6$ and the quantity xy^2 is as large as possible. In view of the restriction $x + y = 6$, the quantity $xy^2 = x(6 - x)^2$. We can therefore try to find a real number x which makes the quantity $x(6 - x)^2$ as large as possible. The idea here is to consider the function $f(x) = x(6 - x)^2$ and hope to find a local maximum. We can write $f(x) = 36x - 12x^2 + x^3$, and so $f'(x) = 36 - 24x + 3x^2 = 3(x^2 - 8x + 12) = 3(x - 2)(x - 6)$. Hence $x = 2$ and $x = 6$ are stationary points. Next, note that $f''(x) = 6x - 24$. Hence $f''(2) = -12$ and $f''(6) = 12$. It follows that the function $f(x)$ has a local maximum at the point $x = 2$. Then $y = 6 - x = 4$, with $f(2) = 32$. This choice of x and y makes xy^2 as large as possible, with value $f(2) = 32$.

EXAMPLE 13.2.2. We have 20 metres of fencing material, and wish to find the largest rectangular area that we can enclose. Suppose that the rectangular area has sides x and y in metres. Then the area is equal to xy, while the perimeter is equal to $2x + 2y$. Hence we wish to maximize the quantity xy subject to the restriction $2x + 2y = 20$. Under the restriction $2x + 2y = 20$, the quantity $xy = x(10 - x)$. We can therefore try to find a real number x which makes the quantity $x(10 - x)$ as large as possible. Consider the function $f(x) = x(10 - x) = 10x - x^2$. Then $f'(x) = 10 - 2x$, and so $x = 5$ is a stationary point. Since $f''(x) = -2$, the point $x = 5$ is a local maximum. Then $y = 10 - x = 5$, with $f(5) = 25$. This choice of x and y makes xy as large as possible, with area 25 square metres.

EXAMPLE 13.2.3. We have 1200 metres of fencing material, and wish to enclose a double paddock with two equal rectangular areas as shown in the diagram below.



Suppose that each of the two rectangular areas has sides x and y in metres, as shown in the picture. Then the total area is equal to $2xy$, while the total perimeter is equal to $4x + 3y$. Hence we wish to maximize the quantity $2xy$ subject to the restriction $4x + 3y = 1200$. Under the restriction $4x + 3y = 1200$, the quantity

$$2xy = 2x\left(400 - \frac{4x}{3}\right).$$

We can therefore try to find a real number x which makes this quantity as large as possible. Consider the function

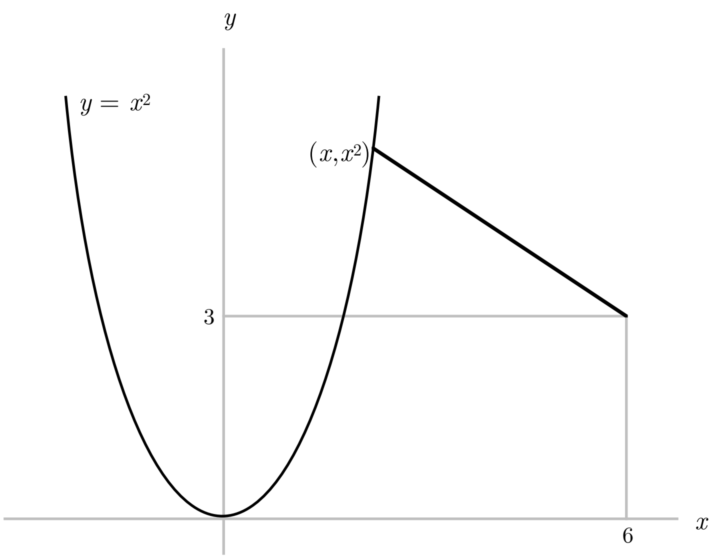
$$f(x) = 2x\left(400 - \frac{4x}{3}\right) = 800x - \frac{8x^2}{3}.$$

Then

$$f'(x) = 800 - \frac{16x}{3},$$

and so $x = 150$ is a stationary point. Since $f''(x) = -16/3$, the point $x = 150$ is a local maximum. Then $y = 200$, with $f(150) = 60000$. This choice of x and y makes $2xy$ as large as possible, with total area 60000 square metres.

EXAMPLE 13.2.4. We wish to find the point on the parabola $y = x^2$ which is closest to the point $(6, 3)$. We begin by drawing a picture.



Note that a typical point on the parabola $y = x^2$ is given by $(x, y) = (x, x^2)$. The distance between this point and the point $(6, 3)$ is given by

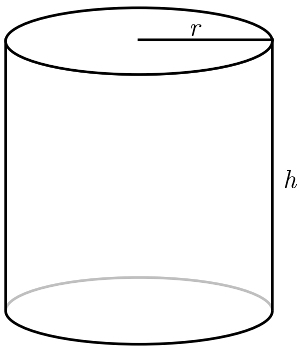
$$\sqrt{(x-6)^2 + (x^2-3)^2},$$

in view of the theorem of Pythagoras. Now let $f(x) = (x-6)^2 + (x^2-3)^2$. Then $f(x)$ represents the square of this distance. We now need to find a local minimum for the function $f(x)$. Differentiating, we obtain $f'(x) = 2(x-6)+4x(x^2-3) = 4x^3-10x-12$. We observe that $x = 2$ is a root of the polynomial $4x^3 - 10x - 12$. Hence we have

$$4x^3 - 10x - 12 = (x-2)(4x^2+8x+6) = (x-2)(4(x^2+2x+1)+2) = (x-2)(4(x+1)^2+2).$$

It follows that there is only one stationary point $x = 2$. Note next that $f''(x) = 12x^2 - 10$, so that $f''(2) > 0$. Hence $x = 2$ is a local minimum. It follows that the point $(2, 4)$ on the parabola is closest to the point $(6, 3)$, with distance $\sqrt{(2-6)^2 + (4-3)^2} = \sqrt{17}$.

EXAMPLE 13.2.5. A manufacturer wishes to maximize the volume of cylindrical metal cans made out of a fixed quantity of metal. To understand this problem, suppose that a typical can has radius r and height h as shown in the picture below:



Then the total surface area is equal to $2\pi r^2 + 2\pi rh = S$, where S is fixed, so that

$$h = \frac{S}{2\pi r} - r. \tag{1}$$

On the other hand, the volume of such a can is equal to $V = \pi r^2 h$. Under the restriction (1), we have

$$V = \pi r^2 h = \frac{Sr}{2} - \pi r^3.$$

Consider now the function

$$V(r) = \frac{Sr}{2} - \pi r^3.$$

Differentiating, we have

$$V'(r) = \frac{S}{2} - 3\pi r^2,$$

so that $r = \sqrt{S/6\pi}$ is the only stationary point, since negative values of r are meaningless. Furthermore, we have $V''(r) = -6\pi r$, and so this stationary point is a local maximum. For this value of r, we have

$$h = \frac{S}{2\pi r} - r = \sqrt{\frac{3S}{2\pi}} - \sqrt{\frac{S}{6\pi}} = \sqrt{\frac{9S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 3\sqrt{\frac{S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 2\sqrt{\frac{S}{6\pi}} = 2r.$$

This means that the most economical shape of a cylindrical can is when the height is twice the radius.

EXAMPLE 13.2.6. A steamer travelling at constant speed due east passes a buoy at 9 am. A hydrofoil travelling at twice this speed due north passes the same buoy at 11 am. We would like to determine the time when the distance between the two vessels is smallest. To set up the problem mathematically, we consider the xy-plane, and assume that the position of the buoy is at the origin $(0, 0)$. Let s be the speed of the steamer. Then at t am, the position of the steamer is given by the point $(s(t-9), 0)$ if we relate due east to the positive horizontal axis. Furthermore, the position of the hydrofoil is given by the point $(0, 2s(t-11))$ if we relate due north to the positive vertical axis. By the theorem of Pythagoras, the distance between the two vessels at t am is given by $\sqrt{s^2(t-9)^2 + 4s^2(t-11)^2}$. Consider now the function $f(t) = s^2(t-9)^2 + 4s^2(t-11)^2$. Clearly this represents the square of the distance between the two vessels, so we need to find a local minimum for this function. Differentiating, we obtain

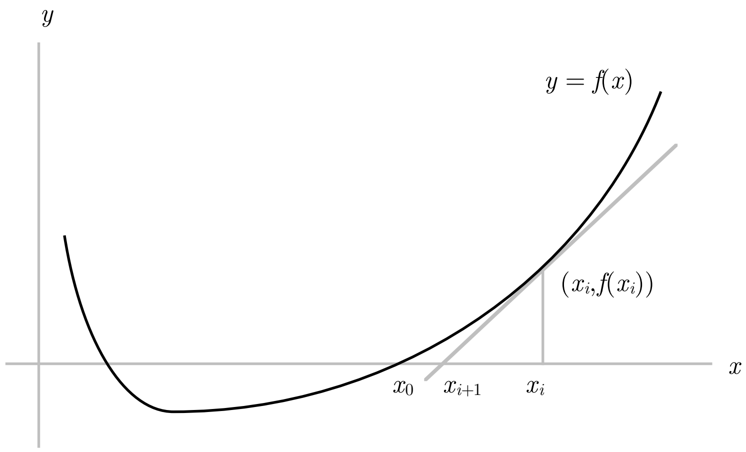
$$f'(t) = 2s^2(t-9) + 8s^2(t-11) = s^2(10t-106),$$

with stationary point $t = 10.6$, representing the time 10:36 am. Can you convince yourself that this is a local minimum?

13.3. Newton's Method

In this section, we briefly describe a numerical technique which allows us to obtain approximations to solutions of some problems where exact answers may be hard or even impossible to calculate.

Let us consider an equation of the form $f(x) = 0$. Suppose that we wish to find some real number x for which the equation is satisfied. Consider the picture below:



Here x_0 represents a solution of the equation $f(x) = 0$. Unfortunately, we are unable to calculate the value of x_0 precisely. We now take some number x_i close to x_0, and consider the tangent to the curve $y = f(x)$ at the point $(x_i, f(x_i))$. Clearly the tangent has slope $f'(x_i)$. It follows that the equation of the tangent is given by

$$\frac{y - f(x_i)}{x - x_i} = f'(x_i).$$

Let x_{i+1} be the x-intercept of this tangent line. Then it is easy to see that

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \qquad (2)$$

From the picture, this new point x_{i+1} gives a better approximation to x_0 than the point x_i does.

Newton's method is now to start with some point x_1 close to a solution x_0 of $f(x) = 0$, and then obtain a sequence of successive approximations $x_2, x_3, x_4, \ldots$ by using the formula (2).

Example 13.3.1. We shall try to obtain some approximation for $\sqrt{2}$. To do so, we consider the equation $f(x) = 0$, where $f(x) = x^2 - 2$. Then $f'(x) = 2x$, and so equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i^2 - 2}{2x_i}.$$

Using this, and taking $x_1 = 2$, we obtain $x_2 = 1.5$, $x_3 = 1.41667$, $x_4 = 1.41422$, $x_5 = 1.41421$, and so on, all to 5 decimal places. On the other hand, if we take $x_1 = -2$, then we obtain $x_2 = -1.5$, $x_3 = -1.41667$, $x_4 = -1.41422$, $x_5 = -1.41421$, and so on, all to 5 decimal places.

Example 13.3.2. We shall try to obtain some approximation to a solution of the equation $x^3+x-1 = 0$. To do so, we consider the function $f(x) = x^3+x-1$. Then $f'(x) = 3x^2+1$, and so equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1}.$$

Using this, and taking $x_1 = 1$, we obtain $x_2 = 0.75$, $x_3 = 0.68605$, $x_4 = 0.68234$, $x_5 = 0.68233$, and so on, all to 5 decimal places.

Remark. As is the case for much of numerical mathematics, Newton's method is imprecise. It may fail to work in some instances. If there are many possible solutions, then it is sometimes unclear which solution the method will give.

Example 13.3.3. Let us return to Example 13.1.7, and consider the function

$$f(x) = \frac{x}{x^2+1}, \qquad \text{with} \qquad f'(x) = \frac{1-x^2}{(x^2+1)^2}.$$

It is easy to see that the equation $f(x) = 0$ has precisely one solution, namely $x = 0$. Nevertheless, let us apply Newton's method to this function. Then equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i(x_i^2+1)}{1-x_i^2}.$$

Using this, and taking $x_1 = 0.5$, we obtain $x_2 = -0.33333$, $x_3 = 0.08333$, $x_4 = -0.00117$, $x_5 = 0.00000$, and so on, all to 5 decimal places. On the other hand, if we take $x_1 = 2$, then we obtain $x_2 = 5.33333$, $x_3 = 11.05533$, $x_4 = 22.29306$, $x_5 = 44.67602$, and so on, all to 5 decimal places. The method clearly fails in this second case.

Problems for Chapter 13

1. For each of the following functions, find all of the stationary points. For each such stationary point, determine whether it is a local maximum, a local minimum or another type of stationary point:
   a) $f(x) = 3x^2 + 6x + 9$
   b) $f(x) = 6x - x^2$
   c) $f(x) = 2 - 3x - 3x^2$
   d) $f(x) = 6 + 9x - 3x^2 - x^3$
   e) $f(x) = x + \dfrac{4}{x+1}$
   f) $f(x) = 4x - 1 + \dfrac{36}{x-1}$
   g) $f(x) = (x+1)^2 - (x-1)^2$
   h) $f(x) = 6 - \dfrac{2}{x} - x^2$
2. A bullet is shot upwards at time $t = 0$ from the top of a building 176 metres tall, with an initial speed of 160 metres per second. The height of the bullet is given by $h(t) = -16t^2 + 160t + 176$ after t seconds. At what time is the bullet at maximum height above the ground? What is this height?
3. What number, when squared and added to 16 times its reciprocal, gives a minimum value for this sum?

4. A piece of wire is to be cut into two pieces to form a circle and a square. How should the wire be cut to minimize the total area of the two pieces?

5. Find two real numbers whose sum is 16 and whose product is a maximum.

6. Find two positive real numbers whose product is 81 and the sum of whose squares is a minimum.

7. What positive real number is exceeded by its square root by the greatest amount?

8. Find the dimension of a right circular cylinder of volume 1 cubic metre and having the minimum surface area.

9. A closed box is to be constructed with a square base and volume of 1500 cubic metres. The material used for the base costs twice as much as for the top and sides. What dimension should the box have to keep the cost to a minimum?

10. Find the maximum area of a rectangle inscribed in a semicircle of radius 12 metres.

11. A rectangular beam, of width w and depth d, is cut from a circular log of diameter $a = 25$ centimetres. The beam has strength S given by $S = 2wd^2$. Find the dimension that will give the strongest beam.
[HINT: Use $d^2 + w^2 = a^2$ to relate the variables d and w.]

12. For each of the following, use a calculator and Newton's method to obtain estimates for the desired quantity to at least 4 decimal places:
    a) The number $\sqrt{5}$, starting with an initial estimate of $x_0 = 2$.
    b) The number $\sqrt{7}$, starting with an initial estimate of $x_0 = 3$.
    c) The number $\sqrt[3]{2}$.
    d) A solution to the equation $x^3 - 2x - 5 = 0$.
    e) A solution to the equation $\cos x = x$.
    f) The largest real root of the polynomial $x^3 + x - 1$.
    g) The largest solution to the equation $\sin x = \mathrm{e}^x$.

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