ELEMENTARY MATHEMATICS

W W L CHEN and X T DUONG

Chapter 4

INDICES AND LOGARITHMS

4.1. Indices

Given any non-zero real number $a \in \mathbb{R}$ and any natural number $k \in \mathbb{N}$, we often write

$$a^k = \underbrace{a \times \ldots \times a}_{k}. \tag{1}$$

This definition can be extended to all integers $k \in \mathbb{Z}$ by writing

$$a^0 = 1 \tag{2}$$

and

$$a^k = \frac{1}{a^{-k}} = \frac{1}{\underbrace{a \times \ldots \times a}_{-k}} \tag{3}$$

whenever k is a negative integer, noting that $-k \in \mathbb{N}$ in this case.

It is not too difficult to establish the following.

LAWS OF INTEGER INDICES. *Suppose that $a, b \in \mathbb{R}$ are non-zero. Then for every $m, n \in \mathbb{Z}$, we have*
(a) $a^m a^n = a^{m+n}$;
(b) $\dfrac{a^m}{a^n} = a^{m-n}$;
(c) $(a^m)^n = a^{mn}$; *and*
(d) $(ab)^m = a^m b^m$.

† This chapter was written at Macquarie University in 1999.

We now further extend the definition of a^k to all rational numbers $k \in \mathbb{Q}$. To do this, we first of all need to discuss the q-th roots of a positive real number a, where $q \in \mathbb{N}$. This is an extension of the idea of square roots discussed in Section 1.2. We recall the following definition, slightly modified here.

DEFINITION. Suppose that $a \in \mathbb{R}$ is positive. We say that $x > 0$ is the positive square root of a if $x^2 = a$. In this case, we write $x = \sqrt{a}$.

We now make the following natural extension.

DEFINITION. Suppose that $a \in \mathbb{R}$ is positive and $q \in \mathbb{N}$. We say that $x > 0$ is the positive q-th root of a if $x^q = a$. In this case, we write $x = \sqrt[q]{a} = a^{1/q}$.

Recall now that every rational number $k \in \mathbb{Q}$ can be written in the form $k = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We may, if we wish, assume that p/q is in lowest terms, where p and q have no common factors. For any positive real number $a \in \mathbb{R}$, we can now define a^k by writing

$$a^k = a^{p/q} = (a^{1/q})^p = (a^p)^{1/q}. \tag{4}$$

In other words, we first of all calculate the positive q-th root of a, and then take the p-th power of this q-th root. Alternatively, we can first of all take the p-th power of a, and then calculate the positive q-th root of this p-th power.

We can establish the following generalization of the Laws of integer indices.

LAWS OF INDICES. *Suppose that $a, b \in \mathbb{R}$ are positive. Then for every $m, n \in \mathbb{Q}$, we have*
(a) $a^m a^n = a^{m+n}$;
(b) $\dfrac{a^m}{a^n} = a^{m-n}$;
(c) $(a^m)^n = a^{mn}$; *and*
(d) $(ab)^m = a^m b^m$.

REMARKS. (1) Note that we have to make the restriction that the real numbers a and b are positive. If $a = 0$, then a^k is clearly not defined when k is a negative integer. If $a < 0$, then we will have problems taking square roots.

(2) It is possible to define cube roots of a negative real number a. It is a real number x satisfying the requirement $x^3 = a$. Note that $x < 0$ in this case. A similar argument applies to q-th roots when $q \in \mathbb{N}$ is odd. However, if $q \in \mathbb{N}$ is even, then $x^q \geq 0$ for every $x \in \mathbb{R}$, and so $x^q \neq a$ for any negative $a \in \mathbb{R}$. Hence a negative real number does not have real q-th roots for any even $q \in \mathbb{N}$.

EXAMPLE 4.1.1. We have $2^4 \times 2^3 = 16 \times 8 = 128 = 2^7 = 2^{4+3}$ and $2^{-3} = 1/8$.

EXAMPLE 4.1.2. We have $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$. Alternatively, we have $8^{2/3} = (8^2)^{1/3} = 64^{1/3} = 4$.

EXAMPLE 4.1.3. To show that $\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}$, we first of all observe that both sides are positive, and so it suffices to show that the squares of the two sides are equal. Note now that

$$(\sqrt{2} + \sqrt{3})^2 = (\sqrt{2})^2 + 2\sqrt{2}\sqrt{3} + (\sqrt{3})^2 = 5 + 2 \times 2^{1/2}3^{1/2} = 5 + 2(2 \times 3)^{1/2} = 5 + 2\sqrt{6},$$

the square of the left hand side.

EXAMPLE 4.1.4. To show that $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$, it suffices to show that the squares of the two sides are equal. Note now that

$$\begin{aligned}(\sqrt{6} - \sqrt{2})^2 &= (\sqrt{6})^2 - 2\sqrt{6}\sqrt{2} + (\sqrt{2})^2 = 8 - 2 \times 6^{1/2}2^{1/2} = 8 - 2(6 \times 2)^{1/2} \\ &= 8 - 2\sqrt{12} = 8 - 2 \times 2\sqrt{3} = 8 - 4\sqrt{3},\end{aligned}$$

the square of the left hand side.

In the remaining examples in this section, the variables x and y both represent positive real numbers.

Example 4.1.5. We have

$$x^{-1} \times 2x^{1/2} = 2x^{(1/2)-1} = 2x^{-1/2} = \frac{2}{x^{1/2}} = \frac{2}{\sqrt{x}}.$$

Example 4.1.6. We have

$$\begin{aligned}(27x^2)^{1/3} \div \frac{1}{3}(x^5)^{1/2} &= 27^{1/3}(x^2)^{1/3} \div \frac{1}{3}x^{5\times(1/2)} = 3x^{2/3} \div \frac{x^{5/2}}{3} = 3x^{2/3} \times \frac{3}{x^{5/2}} \\ &= 3x^{2/3} \times 3x^{-5/2} = 9x^{2/3}x^{-5/2} = 9x^{(2/3)-(5/2)} = 9x^{-11/6} = \frac{9}{x^{11/6}}.\end{aligned}$$

Example 4.1.7. We have

$$(9x)^{1/2}(8x^{-1/2})^{1/3} = 9^{1/2}x^{1/2} \times 8^{1/3}x^{(-1/2)\times(1/3)} = 3x^{1/2} \times 2x^{-1/6} = 6x^{(1/2)-(1/6)} = 6x^{1/3}.$$

Example 4.1.8. We have

$$\begin{aligned}(8x^{3/4})^{-2} \div \left(\frac{1}{2}x^{-1}\right)^2 &= (8x^{3/4})^{-2} \times \left(\frac{1}{2}x^{-1}\right)^{-2} = 8^{-2}x^{(3/4)\times(-2)} \times \left(\frac{1}{2}\right)^{-2} x^{(-1)\times(-2)} \\ &= \frac{1}{8^2}x^{-3/2} \times 2^2x^2 = \frac{1}{16}x^{(-3/2)+2} = \frac{1}{16}x^{1/2} = \frac{\sqrt{x}}{16}.\end{aligned}$$

Example 4.1.9. We have

$$\begin{aligned}\sqrt[3]{8a^2b} \times a^{1/3}b^{5/3} &= (8a^2b)^{1/3}a^{1/3}b^{5/3} = 8^{1/3}(a^2)^{1/3}b^{1/3}a^{1/3}b^{5/3} = 2a^{2/3}b^{1/3}a^{1/3}b^{5/3} \\ &= 2a^{(2/3)+(1/3)}b^{(1/3)+(5/3)} = 2ab^2.\end{aligned}$$

Example 4.1.10. We have

$$\begin{aligned}\sqrt[4]{(16x^{1/6}y^2)^3} &= ((16x^{1/6}y^2)^3)^{1/4} = (16x^{1/6}y^2)^{3/4} = 16^{3/4}(x^{1/6})^{3/4}(y^2)^{3/4} \\ &= (16^{1/4})^3x^{(1/6)\times(3/4)}y^{2\times(3/4)} = 8x^{1/8}y^{3/2}.\end{aligned}$$

We now miss out some intermediate steps in the examples below. The reader is advised to fill in all the details in each step.

Example 4.1.11. We have

$$\begin{aligned}\frac{16(x^2y^3)^{1/2}}{(2x^{1/2}y)^3} \times \frac{(4x^6y^4)^{1/2}}{(6x^3y^{1/2})^2} &= \frac{16xy^{3/2}}{8x^{3/2}y^3} \times \frac{2x^3y^2}{36x^6y} = \frac{16xy^{3/2} \times 2x^3y^2}{8x^{3/2}y^3 \times 36x^6y} = \frac{32x^4y^{7/2}}{288x^{15/2}y^4} \\ &= \frac{32}{288}x^4y^{7/2}x^{-15/2}y^{-4} = \frac{1}{9}x^{-7/2}y^{-1/2} = \frac{1}{9x^{7/2}y^{1/2}}.\end{aligned}$$

Example 4.1.12. We have

$$\begin{aligned}\frac{(3x)^3y^2}{(5xy)^2} \div \frac{(5xy)^4}{(27x^9)^3} &= \frac{(3x)^3y^2}{(5xy)^2} \times \frac{(27x^9)^3}{(5xy)^4} = \frac{(3x)^3y^2 \times (27x^9)^3}{(5xy)^2 \times (5xy)^4} = \frac{3^3 27^3 x^{30}y^2}{5^6x^6y^6} \\ &= \frac{531441}{15625}x^{30}y^2x^{-6}y^{-6} = \frac{531441}{15625}x^{24}y^{-4} = \frac{531441x^{24}}{15625y^4}.\end{aligned}$$

Example 4.1.13. We have

$$\begin{aligned}\frac{x^{-1}+y^{-1}}{x+y}-\frac{x^{-1}-y^{-1}}{x-y}&=\frac{(x-y)(x^{-1}+y^{-1})-(x^{-1}-y^{-1})(x+y)}{(x-y)(x+y)}\\&=\frac{(1+xy^{-1}-yx^{-1}-1)-(1+yx^{-1}-xy^{-1}-1)}{(x-y)(x+y)}=\frac{2(xy^{-1}-yx^{-1})}{(x-y)(x+y)}\\&=\frac{2}{(x-y)(x+y)}\times\left(\frac{x}{y}-\frac{y}{x}\right)=\frac{2}{(x-y)(x+y)}\times\frac{x^2-y^2}{xy}=\frac{2}{xy}.\end{aligned}$$

Example 4.1.14. We have

$$\begin{aligned}\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}&=(x^{-1}-y^{-1})\div(x^{-2}-y^{-2})=\left(\frac{1}{x}-\frac{1}{y}\right)\div\left(\frac{1}{x^2}-\frac{1}{y^2}\right)=\frac{y-x}{xy}\div\frac{y^2-x^2}{x^2y^2}\\&=\frac{y-x}{xy}\times\frac{x^2y^2}{y^2-x^2}=\frac{x^2y^2(y-x)}{xy(y^2-x^2)}=\frac{x^2y^2(y-x)}{xy(y-x)(y+x)}=\frac{xy}{x+y}.\end{aligned}$$

Alternatively, write $a=x^{-1}$ and $b=y^{-1}$. Then

$$\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}=\frac{a-b}{a^2-b^2}=\frac{1}{a+b}=(a+b)^{-1}=\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}=\left(\frac{x+y}{xy}\right)^{-1}=\frac{xy}{x+y}.$$

Example 4.1.15. We have

$$x^2y^{-1}\div(x^{-1}+y^{-1})=\frac{x^2}{y}\div\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{x^2}{y}\div\frac{x+y}{xy}=\frac{x^2}{y}\times\frac{xy}{x+y}=\frac{x^3}{x+y}.$$

Example 4.1.16. We have

$$\begin{aligned}xy\div((x^{-1}+y)^{-1})^{-1}&=xy\times(x^{-1}+y)^{-1}=xy\times\left(\frac{1}{x}+y\right)^{-1}\\&=xy\times\left(\frac{1+xy}{x}\right)^{-1}=xy\times\frac{x}{1+xy}=\frac{x^2y}{1+xy}.\end{aligned}$$

4.2. The Exponential Functions

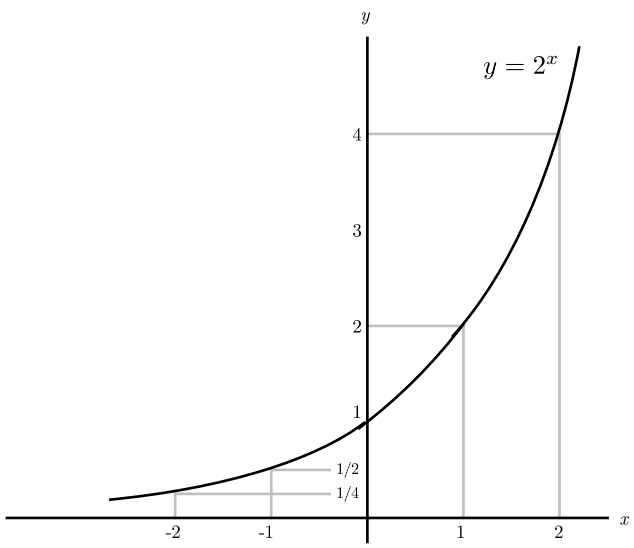
Suppose that $a\in\mathbb{R}$ is a positive real number. We have shown in Section 4.1 that we can use (1)–(4) to define a^k for every rational number $k\in\mathbb{Q}$. Here we shall briefly discuss how we may further extend the definition of a^k to a function a^x defined for every real number $x\in\mathbb{R}$. A thorough treatment of this extension will require the study of the theory of continuous functions as well as the well known result that the rational numbers are "dense" among the real numbers, and is beyond the scope of this set of notes. We shall instead confine our discussion here to a heuristic treatment.

We all know simple functions like $y=x^2$ (a parabola) or $y=2x+3$ (a straight line). It is possible to draw the graph of such a function in one single stroke, without lifting our pen from the paper before completing the drawing. Such functions are called continuous functions.

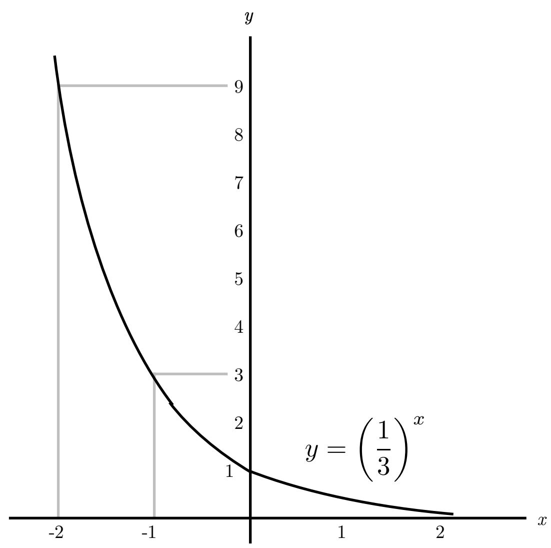
Suppose now that a positive real number $a\in\mathbb{R}$ has been chosen and fixed. Note that we have already defined a^k for every rational number $k\in\mathbb{Q}$. We now draw the graph of a continuous function on the xy-plane which will pass through every point (k,a^k) where $k\in\mathbb{Q}$. It turns out that such a function is unique. In other words, there is one and only one continuous function whose graph on the xy-plane will pass through every point (k,a^k) where $k\in\mathbb{Q}$. We call this function the exponential function corresponding to the positive real number a, and write $f(x)=a^x$.

LAWS FOR EXPONENTIAL FUNCTIONS. *Suppose that $a \in \mathbb{R}$ is positive. Then $a^0 = 1$. For every $x_1, x_2 \in \mathbb{R}$, we have*
(a) $a^{x_1} a^{x_2} = a^{x_1+x_2}$;
(b) $\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}$; *and*
(c) $(a^{x_1})^{x_2} = a^{x_1 x_2}$.
(d) Furthermore, if $a \neq 1$, then $a^{x_1} = a^{x_2}$ if and only if $x_1 = x_2$.

EXAMPLE 4.2.1. The graph of the exponential function $y = 2^x$ is shown below:



EXAMPLE 4.2.2. The graph of the exponential function $y = (1/3)^x$ is shown below:



EXAMPLE 4.2.3. We have $8^x \times 2^{4x} = (2^3)^x \times 2^{4x} = 2^{3x}2^{4x} = 2^{7x} = (2^7)^x = 128^x$.

EXAMPLE 4.2.4. We have $9^{x/2} \times 27^{x/3} = (9^{1/2})^x \times (27^{1/3})^x = 3^x 3^x = 3^{2x} = (3^2)^x = 9^x$.

Example 4.2.5. We have

$$32^{x+2} \div 8^{2x-1} = 32^{x+2} \times 8^{1-2x} = (2^5)^{x+2} \times (2^3)^{1-2x} = 2^{5x+10}2^{3-6x} = 2^{13-x}.$$

Example 4.2.6. We have

$$64^{2x} \div 16^{2x} = 2^{12x} \div 2^{8x} = \frac{2^{12x}}{2^{8x}} = 2^{4x} = (2^4)^x = 16^x.$$

Example 4.2.7. We have

$$\frac{5^{x+1} + 5^{x-1}}{5^{x+2} + 5^x} = \frac{5^{x+1} + 5^{x-1}}{5 \times 5^{x+1} + 5 \times 5^{x-1}} = \frac{5^{x+1} + 5^{x-1}}{5(5^{x+1} + 5^{x-1})} = \frac{1}{5}.$$

Example 4.2.8. We have

$$\frac{4^x - 2^{x-1}}{2^x - \frac{1}{2}} = \frac{2^{2x} - 2^{-1}2^x}{2^x - \frac{1}{2}} = \frac{2^x 2^x - \frac{1}{2}2^x}{2^x - \frac{1}{2}} = \frac{2^x(2^x - \frac{1}{2})}{2^x - \frac{1}{2}} = 2^x.$$

Example 4.2.9. Suppose that

$$25^x = \frac{1}{\sqrt{125}}.$$

We can write $25^x = (5^2)^x = 5^{2x}$ and

$$\frac{1}{\sqrt{125}} = (\sqrt{125})^{-1} = ((125)^{1/2})^{-1} = ((5^3)^{1/2})^{-1} = 5^{-3/2}.$$

It follows that we must have $2x = -3/2$, so that $x = -3/4$.

Example 4.2.10. Suppose that

$$\left(\frac{1}{9}\right)^{2x-1} = 3(27^{-x}).$$

We can write $3(27^{-x}) = 3((3^3)^{-x}) = 3 \times 3^{-3x} = 3^{1-3x}$ and

$$\left(\frac{1}{9}\right)^{2x-1} = \left(\frac{1}{3^2}\right)^{2x-1} = (3^{-2})^{2x-1} = 3^{2-4x}.$$

It follows that we must have $1 - 3x = 2 - 4x$, so that $x = 1$.

Example 4.2.11. Suppose that $9^x = \sqrt{3}$. We can write $9^x = (3^2)^x = 3^{2x}$ and $\sqrt{3} = 3^{1/2}$. It follows that we must have $2x = 1/2$, so that $x = 1/4$.

Example 4.2.12. Suppose that $5^{3x-4} = 1$. We can write $1 = 5^0$. It follows that we must have $3x - 4 = 0$, so that $x = 4/3$.

Example 4.2.13. Suppose that $(0.125)^x = \sqrt{0.5}$. We can write

$$(0.125)^x = \left(\frac{1}{8}\right)^x = \left(\frac{1}{2^3}\right)^x = (2^{-3})^x = 2^{-3x}$$

and

$$\sqrt{0.5} = (0.5)^{1/2} = \left(\frac{1}{2}\right)^{1/2} = (2^{-1})^{1/2} = 2^{-1/2}.$$

It follows that we must have $-3x = -1/2$, so that $x = 1/6$.

EXAMPLE 4.2.14. Suppose that $8^{1-x} \times 2^{x-3} = 4$. We can write $4 = 2^2$ and

$$8^{1-x} \times 2^{x-3} = (2^3)^{1-x} \times 2^{x-3} = 2^{3-3x} \times 2^{x-3} = 2^{-2x}.$$

It follows that we must have $-2x = 2$, so that $x = -1$.

4.3. The Logarithmic Functions

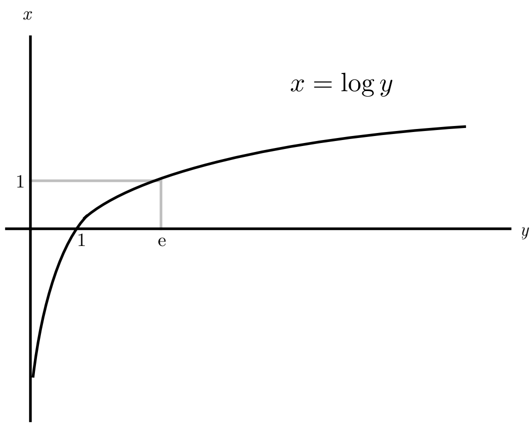
The logarithmic functions are the inverses of the exponential functions. Suppose that $a \in \mathbb{R}$ is a positive real number and $a \neq 1$. If $y = a^x$, then x is called the logarithm of y to the base a, denoted by $x = \log_a y$. In other words, we have

$$y = a^x \qquad \text{if and only if} \qquad x = \log_a y.$$

For the special case when $a = 10$, we have the common logarithm $\log_{10} y$ of y. Another special case is when $a = \mathrm{e} = 2.7182818\ldots$, an irrational number. We have the natural logarithm $\log_{\mathrm{e}} y$ of y, sometimes also denoted by $\log y$ or $\ln y$. Whenever we mention a logarithmic function without specifying its base, we shall assume that it is base e.

REMARK. The choice of the number e is made to ensure that the derivative of the function e^x is also equal to e^x for every $x \in \mathbb{R}$. This is not the case for any other non-zero function, apart from constant multiples of the function e^x.

EXAMPLE 4.3.1. The graph of the logarithmic function $x = \log y$ is shown below:



LAWS FOR LOGARITHMIC FUNCTIONS. *Suppose that $a \in \mathbb{R}$ is positive and $a \neq 1$. Then $\log_a 1 = 0$ and $\log_a a = 1$. For every positive $y_1, y_2 \in \mathbb{R}$, we have*
(a) $\log_a(y_1 y_2) = \log_a y_1 + \log_a y_2$;
(b) $\log_a \left(\dfrac{y_1}{y_2} \right) = \log_a y_1 - \log_a y_2$; *and*
(c) $\log_a(y_1^k) = k \log_a y_1$ *for every* $k \in \mathbb{R}$.
(d) Furthermore, $\log_a y_1 = \log_a y_2$ *if and only if* $y_1 = y_2$.

INVERSE LAWS. *Suppose that $a \in \mathbb{R}$ is positive and $a \neq 1$.*
(a) For every real number $x \in \mathbb{R}$, we have $\log_a(a^x) = x$.
(b) For every positive real number $y \in \mathbb{R}$, we have $a^{\log_a y} = y$.

EXAMPLE 4.3.2. We have $\log_2 8 = x$ if and only if $2^x = 8$, so that $x = 3$. Alternatively, we can use one of the Inverse laws to obtain $\log_2 8 = \log_2(2^3) = 3$.

EXAMPLE 4.3.3. We have $\log_5 125 = x$ if and only if $5^x = 125$, so that $x = 3$.

EXAMPLE 4.3.4. We have $\log_3 81 = x$ if and only if $3^x = 81$, so that $x = 4$.

EXAMPLE 4.3.5. We have $\log_{10} 1000 = x$ if and only if $10^x = 1000$, so that $x = 3$.

EXAMPLE 4.3.6. We have

$$\log_4 \frac{1}{8} = x \qquad \text{if and only if} \qquad 4^x = \frac{1}{8}.$$

This means that $2^{2x} = 2^{-3}$, so that $x = -3/2$.

EXAMPLE 4.3.7. We have

$$\log_2 \frac{\sqrt{2}}{16} = \log_2 \left(\frac{2^{1/2}}{2^4} \right) = \log_2(2^{-7/2}) = -\frac{7}{2}.$$

EXAMPLE 4.3.8. We have

$$\log_3 \frac{1}{3\sqrt{3}} = \log_3(3^{-3/2}) = -\frac{3}{2}.$$

EXAMPLE 4.3.9. Use your calculator to confirm that the following are correct to 3 decimal places:

$$\log_{10} 20 \approx 1.301, \quad \log_{10} 7 \approx 0.845, \quad \log_{10} \frac{1}{5} \approx -0.698,$$
$$\log 5 \approx 1.609, \qquad \log 21 \approx 3.044, \qquad \log 0.2 \approx -1.609.$$

EXAMPLE 4.3.10. We have

$$\log_a \frac{1}{3} = -\frac{1}{3} \qquad \text{if and only if} \qquad a^{-1/3} = \frac{1}{3}.$$

It follows that

$$a = (a^{-1/3})^{-3} = \left(\frac{1}{3} \right)^{-3} = 3^3 = 27.$$

EXAMPLE 4.3.11. We have

$$\log_a \frac{1}{4} = -\frac{2}{3} \qquad \text{if and only if} \qquad a^{-2/3} = \frac{1}{4}.$$

It follows that

$$a = (a^{-2/3})^{-3/2} = \left(\frac{1}{4} \right)^{-3/2} = 4^{3/2} = 8.$$

EXAMPLE 4.3.12. We have

$$\log_a 4 = \frac{1}{2} \qquad \text{if and only if} \qquad a^{1/2} = 4.$$

It follows that $a = 16$.

EXAMPLE 4.3.13. Suppose that $\log_5 y = -2$. Then $y = 5^{-2} = 1/25$.

EXAMPLE 4.3.14. Suppose that $\log_a y = \log_a 3 + \log_a 5$. Then since $\log_a 3 + \log_a 5 = \log_a 15$, we must have $y = 15$.

EXAMPLE 4.3.15. Suppose that $\log_a y + 2\log_a 4 = \log_a 20$. Then

$$\log_a y = \log_a 20 - 2\log_a 4 = \log_a 20 - \log_a(4^2) = \log_a 20 - \log_a 16 = \log_a \frac{20}{16},$$

so that $y = 20/16 = 5/4$.

EXAMPLE 4.3.16. Suppose that

$$\frac{1}{2}\log 6 - \log y = \log 12.$$

Then

$$\log y = \frac{1}{2}\log 6 - \log 12 = \log(\sqrt{6}) - \log 12 = \log \frac{\sqrt{6}}{12},$$

so that $y = \sqrt{6}/12$.

For the next four examples, u and v are positive real numbers.

EXAMPLE 4.3.17. We have $\log_a(u^{10}) \div \log_a u = 10\log_a u \div \log_a u = 10$.

EXAMPLE 4.3.18. We have $5\log_a u - \log_a(u^5) = 5\log_a u - 5\log_a u = 0$.

EXAMPLE 4.3.19. We have

$$2\log_a u + 2\log_a v - \log_a((uv)^2) = \log_a(u^2) + \log_a(v^2) - \log_a(u^2v^2) = \log_a(u^2v^2) - \log_a(u^2v^2) = 0.$$

EXAMPLE 4.3.20. We have

$$\begin{aligned}\left(\log_a \frac{u^3}{v} + \log_a \frac{v}{u}\right) \div \log_a(\sqrt{u}) &= \log_a\left(\frac{u^3}{v} \times \frac{v}{u}\right) \div \log_a(u^{1/2}) \\ &= \log_a(u^2) \div \log_a(u^{1/2}) = 2\log_a u \div \frac{1}{2}\log_a u = 4.\end{aligned}$$

EXAMPLE 4.3.21. Suppose that $2\log y = \log(4 - 3y)$. Then since $2\log y = \log(y^2)$, we must have $y^2 = 4 - 3y$, so that $y^2 + 3y - 4 = 0$. This quadratic equation has roots

$$y = \frac{-3 \pm \sqrt{9+16}}{2} = 1 \text{ or } -4.$$

However, we have to discard the solution $y = -4$, since $\log(-4)$ is not defined. The only solution is therefore $y = 1$.

EXAMPLE 4.3.22. Suppose that $\log(\sqrt{y}) = \sqrt{\log y}$. Then

$$\frac{1}{2}\log y = \sqrt{\log y}, \qquad \text{and so} \qquad \log y = 2\sqrt{\log y}.$$

Squaring both sides and letting $x = \log y$, we obtain the quadratic equation $x^2 = 4x$, with solutions $x = 0$ and $x = 4$. The equation $\log y = 0$ corresponds to $y = 1$. The equation $\log y = 4$ corresponds to $y = \mathrm{e}^4$.

Problems for Chapter 4

1. Simplify each of the following expressions:
   a) $216^{2/3}$ b) $32^{3/5}$ c) $64^{-1/6}$ d) $10000^{-3/4}$

2. Simplify each of the following expressions, where x denotes a positive real number:
   a) $(5x^2)^3 \div (25x^{-4})^{1/2}$ b) $(32x^2)^{2/5} \times \left(\dfrac{25}{x^4}\right)^{1/2}$ c) $(16x^{12})^{3/4} \times \left(\dfrac{27}{x^6}\right)^{-1/3}$
   d) $(128x^{14})^{-1/7} \div \left(\dfrac{1}{9}x^{-1}\right)^{-1/2}$ e) $\dfrac{(2x^3)^{1/2}}{(4x)^2} \div \dfrac{(8x)^{1/2}}{(3x^2)^3}$

3. Simplify each of the following expressions, where x and y denote suitable positive real numbers:
   a) $(x^2y^3)^{1/3}(x^3y^2)^{-1/2}$ b) $\dfrac{x^{-1}-y^{-1}}{x^{-3}-y^{-3}}$
   c) $\dfrac{x^{-2}+y^{-2}}{x+y} - \dfrac{x^{-2}-y^{-2}}{x-y}$ d) $xy \div (x^{-1}+y^{-1})^{-1}$
   e) $(x^{-2}-y^{-2})(x-y)^{-1}\left(\dfrac{1}{xy}\right)^{-1}(x^{-1}+y^{-1})^{-1}$ f) $(36x^{1/2}y^2)^{3/2} \div \left(\dfrac{6x^{-2}}{y^{-2/3}}\right)^3$

4. Determine whether each of the following statements is correct:
   a) $\sqrt{8+2\sqrt{15}} = \sqrt{3}+\sqrt{5}$ b) $\sqrt{16-4\sqrt{15}} = \sqrt{6}-\sqrt{10}$

5. Simplify each of the following expressions, where x denotes a positive real number:
   a) $32^x \times 2^{3x}$ b) $16^{3x/4} \div 4^{2x}$ c) $\dfrac{2^{x+1}+4^x}{2^{x-1}+1}$ d) $\dfrac{9^x-4^x}{3^x+2^x}$

6. Solve each of the following equations:
   a) $27^{2-x} = 9^{x-2}$ b) $4^x = \dfrac{1}{\sqrt{32}}$ c) $\left(\dfrac{1}{9}\right)^x = 3 \times 81^{-x}$
   d) $2^x \times 16^x = 4 \times 8^x$ e) $4^x = 3^x$ f) $\left(\dfrac{1}{7}\right)^x = 7^{2x}$

7. Find the precise value of each of the following expressions:
   a) $\log_3 \dfrac{1}{27}$ b) $\log_{100} 1000$ c) $\log_5 \sqrt{125}$ d) $\log_4 \dfrac{1}{32}$

8. Solve each of the following equations:
   a) $\log_2 y + 3\log_2(2y) = 3$ b) $\log_3 y = -3$ c) $\log 7 - 2\log y = 2\log 49$
   d) $2\log y = \log(y+2)$ e) $2\log y = \log(7y-12)$ f) $\log y = 3\sqrt{\log y}$
   g) $2\log y = \log(y+6)$ h) $2\log y = \sqrt{\log y}$

9. Simplify each of the following expressions, where u, v and w denote suitable positive real numbers:
   a) $\log \dfrac{u^4}{v^2} - \log \dfrac{v^2}{u}$ b) $\log(u^3v^{-6}) - 3\log \dfrac{u}{v^2}$
   c) $3\log u - \log(uv)^3 + 3\log(vw)$ d) $\log \dfrac{uv}{w} + \log \dfrac{uw}{v} + \log \dfrac{vw}{u} - \log(uvw)$
   e) $\left(\log \dfrac{u^4}{v^2} - \log v + 2\log u\right) \div (2\log u - \log v)$ f) $\log(u^2-v^2) - \log(u-v) - \log(u+v)$
   g) $\log u^3 + \log v^2 - \log\left(\dfrac{u}{v}\right)^3 - 5\log v$ h) $2\log(\mathrm{e}^u) + \log(\mathrm{e}^v) - \log(\mathrm{e}^{u+v})$

− ∗ − ∗ − ∗ − ∗ − ∗ −