PART I - BASICS

Exercise 1. Calculate derivatives in given points using the definition of a derivative.

a) $f(x) = x^4,\ x_0 = 2$, b) $f(x) = \cos x,\ x_0 = \frac{\pi}{4}$, c) $f(x) = x^3 + x^2 + x,\ x_0 \in \mathbf{R}$.

Exercise 2. Calculate the following derivatives.

a) $(x^{10} + x^9 + \cdots + x^2 + x + 1)'$, b) $(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3})'$, c) $(\sin x + \cos x + \tan x + \cot x)'$,
d) $(\ln x + \log_2 x + \log x)'$, e) $(2^x - 3^x + e^x + e)'$, f) $(\arctan x + \arcsin x)'$,
g) $(\sqrt[5]{x} - \sqrt[4]{x} + \sqrt[3]{x} - \sqrt{x})'$, h) $(\sqrt[n]{x})',\ n \in \mathbf{N}$ i) $(\sqrt{\sqrt{x}})'$,
j) $(\sqrt{x} \cdot \cos x)'$, k) $(\log x \cdot \sin x)'$, l) $(\ln x \cdot \arctan x)'$,
m) $((x^2 + 2x + 1) \cdot (\cos x + \sin x))'$, n) $(\arctan x \cdot \operatorname{arccot} x)'$, o) $(\tan x \cdot \cot x)'$,
p) $(\frac{\ln x}{2^x})'$, q) $(\frac{2^x}{\ln x})'$, r) $(\frac{\sqrt{x}}{\arctan x})'$,
s) $(\frac{2x^2+3x^3}{4x^2+x+1})'$, t) $(\frac{\sin x}{\cos x})'$, u) $(\frac{e^x}{2^x})'$.

Exercise 3. Calculate the following derivatives using the chain rule.

a) $(\sqrt{3x+5})'$, b) $(\cos(4x+2))'$, c) $(\sin(\sqrt{x}))'$, d) $(\arctan(\pi x))'$,
e) $(\cos(\sin x))'$, f) $(\ln(2x^2+6x+4))'$, g) $(\tan(3x))'$, h) $(\tan^2(3x))'$,
i) $(e^{5x} \cdot x^2)'$, j) $(2^{\sin x})'$, k) $(\sin^2 x)'$, l) $(\arctan(\sin x))'$,
m) $(\sin(\arctan x))'$, n) $(\ln(\sin x + \cos x))'$, o) $(\sin(\cos(\sin x)))'$, p) $(\sin(\ln(2x+5)))'$.

Exercise 4. Calculate derivatives of the following functions and set their domains.

a) $f(x) = \sqrt[4]{x}$, b) $f(x) = x\sqrt{x} + 2\sqrt[3]{x}$, c) $f(x) = \log(\log x)$.

Exercise 5. Calculate derivatives of the second order.

a) $(x^n)'',\ n \in \mathbf{N}$, b) $(\sin x)''$, c) $(\sqrt{x})''$, d) $(\ln x)''$,
e) $(e^x \cos x)''$, f) $(\frac{x^2}{x-2})''$, g) $(\sin x \cos x)''$, h) $(\frac{1-\cos x}{\sin x})''$.

Exercise 6. Calculate derivatives of given orders.

a) $\sin^{(n)} x,\ n \in \mathbf{N}$, b) $\log^{(3)} x$, c) $(5x^5 + 6x^4 + 7x^3)^{(6)}$, d) $(\frac{1}{x})^{(t)},\ t \in \{2,3,4,5\}$.

Exercise 7. Formulate tangent and normal lines equations at given points.

a) $f(x) = \frac{x-2}{x+3},\ x_0 = -2$,

b) $f(x) = \arcsin(\frac{1-x}{3}),\ x_0 = 1$,

c) $f(x) = \frac{x-1}{x^2},\ x_0 = -1$.

Exercise 8. Find the approximated value using total differential. Compare your result with an exact value and caculate errors.

a) $\sqrt{3 + \frac{1}{0{,}98}}$, b) $\ln(9 - (2.01)^3)$, c) $(45.3)^2$.

Exercise 9. Calculate derivatives of the following functions.

a) $f(x) = x^2 \cdot y + y^2 + 2xy$, b) $f(y) = x^2 \cdot y + y^2 + 2xy$, c) $f(x) = \cos x \sin y$.

Exercise 10. Calculate the following limits. In each case check whether the assumptions of l'Hospitals theorem are met.

$$\text{a) } \lim_{x\to 0}\frac{x-\arctan x}{x^3}, \quad \text{b) } \lim_{x\to 0^+}\frac{\ln(\sin 2x)}{\ln(\sin x)}, \quad \text{c) } \lim_{x\to 0}(\cot x - \tfrac{1}{x}),$$
$$\text{d) } \lim_{x\to 0}(\tfrac{2}{\pi}\arctan x)^{x^2}, \quad \text{e) } \lim_{x\to\infty} x\cdot(e^{\frac{1}{x}}-1), \quad \text{f) } \lim_{x\to 3^+}(x-3)\cdot e^{\frac{1}{x-3}}.$$

PART II - APPLICATIONS

Exercise 11. Find extremes of the following functions and determine monotonicity.

$$a(x)=\arctan x-\ln x, \quad b(x)=\tfrac{x^2+1}{x}, \quad c(x)=\tfrac{x}{x^2+1}, \quad d(x)=3x^4-4x^3-6x^2+12x+4,$$
$$e(x)=\tfrac{x}{\ln x}, \quad f(x)=x\sqrt{x-4}, \quad g(x)=x^2e^{-x}, \quad h(x)=\ln 1+x^2.$$

Exercise 12. Find the largest and the smallest values of the following functions in given intervals.

$$a(x)=\tfrac{2}{\cos x},\ x\in[-\tfrac{\pi}{4},\tfrac{\pi}{4}], \quad b(x)=\tfrac{1}{x^2-1},\ x\in[-\tfrac{1}{2},\tfrac{1}{2}],$$
$$c(x)=x^2\ln x,\ x\in[\tfrac{1}{e},2], \quad d(x)=|x-1|,\ x\in[0,3].$$

Exercise 13. Find local extremes of the following functions.

$$a(x)=\arctan\tfrac{x}{3}-x, \quad b(x)=\ln(x^2-1)+\tfrac{1}{x^2-1}, \quad c(x)=2-|x|, \quad d(x)=\ln^2 x-2\ln x.$$

Exercise 14. Find inflection points and examine concavity of the following functions.

$$a(x)=x^4e^{-x}, \quad b(x)=x\sqrt{x-4}, \quad c(x)=\ln(x^2+1), \quad d(x)=\arctan x-x,$$
$$e(x)=\tfrac{1}{2}(e^x-e^{-x}), \quad f(x)=xe^{-x}, \quad g(x)=\tfrac{\ln(2x)}{x}, \quad h(x)=x\sin(\ln x).$$

Exercise 15. Sketch an example of a graph of:

a) a decreasing function that is concave up to the left and concave down to the right,

b) an increasing function that is concave up to the left and concave down to the right,

c) an increasing function that is concave down to the left and concave up to the right,

d) a function that changes its concavity, but not its monotonicity,

e) a function that changes its monotonicity, but not its concavity.

All functions need to be continuous and differentiable in $\mathbf{R}$.

Exercise 16. Find all asymptotes of the following function graphs.

$$\text{a) } f(x)=\tfrac{x^3}{x^2+1}, \quad \text{b) } f(x)=\tfrac{x^3}{3(x^2-x-2)}, \quad \text{c) } f(x)=\arctan\tfrac{x}{3}-x.$$

Exercise 17. Find domain, co-domain, monotonicity, extremes, inflection points, concavity and asymptotes of the following functions and sketch their graphs.

$$a(x)=\tfrac{x^2}{2(x-3)}, \quad b(x)=\tfrac{x}{\sqrt{x^2-4}}, \quad c(x)=x^2\ln x, \quad d(x)=xe^{-x},$$
$$e(x)=\tfrac{x}{\ln x}, \quad f(x)=x^2e^{-x}, \quad g(x)=x\sqrt{x-4}, \quad h(x)=\ln(1+x^2).$$

Some exercises were taken from the script "Matematyka - podstawy z elementami matematyki wyższej" issued by the Gdańsk University of Technology publishing house.