Increasing and Decreasing Functions

A function $y = f(x)$ is said to increase throughout an interval A if y increases as x increases. That is, whenever $x_2 > x_1$ in A, we find $f(x_2) > f(x_1)$. Similarly, $y = f(x)$ decreases throughout A if y decreases as x increases. Increase is associated with positive derivatives and decrease with negative derivatives.

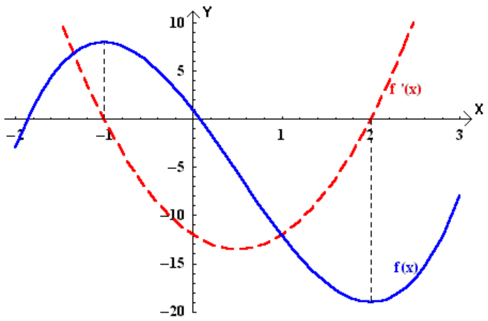
The First Derivative Test for Rise and Fall

Suppose that a function f has a derivative at every poin x of an interval A. Then

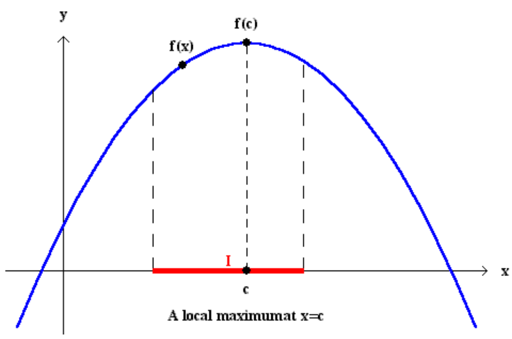
- f increases on A if $f'(x) > 0$ for all x in A,
- f decreases on A if $f'(x) < 0$ for all x in A.

Example

Consider the function $f(x) = 2x^3 - 3x^2 - 12x + 1$. $f(x)$ is a polynomial function. Therefore it is continuous and differentiable everywhere.



Taking the derivative we get $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$.
From the graph, we see that the points $x = -1$ and $x = 2$ are "special". Indeed, at $x = -1$ the function behaves like a point at the top of a hill while at $x = 2$ the graph looks like a valley.
In geometric terms, the first derivative test says that differentable function increase in intervals where their graphs have positive slopes and decrease in intervals where their graphs have negative slopes.



We will discuss the occurrence of local maxima and local minima of a function.

Local Extremum

Let f be a function defined on a domain D, and c a point in D.

If there exists a neighborhood U of c with $f(c) > f(x)$ for all x in U, then $f(c)$ is called a local maximum for the function f that occurs at $x = c$.

If there exists a neighborhood U of c with $f(c) < f(x)$ for all x in U, then $f(c)$ is called a local minimum for the function f that occurs at $x = c$.

Remark

If $f(x)$ has either a local minimum or a local maximum at $x = c$, then $f(c)$ is called local extremum of the function f.

On a graph of a function, its local maxima will look like the tops of hills and its local minima will look like the bottoms of valleys.

Theorem

If $f(x)$ has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.

These points are called critical points.

First Derivative Test

If c is a critical point for $f(x)$, such that $f'(x)$ changes its sign as x crosses from the left to the right of c, then c is a local extremum.

Second-Derivative Test

Let c be a critical point for $f(x)$ such that $f'(c) = 0$.

- If $f''(c) > 0$, then $f'(x)$ is increasing in an interval around c. Since $f'(c) = 0$, then $f'(x)$ must be negative to the left of c and positive to the right of c. Therefore, c is a local minimum.

- If $f''(c) < 0$, then $f'(x)$ is decreasing in an interval around c. Since $f'(c) = 0$, then $f'(x)$ must be positive to the left of c and negative to the right of c. Therefore, c is a local maximum.

THEOREM

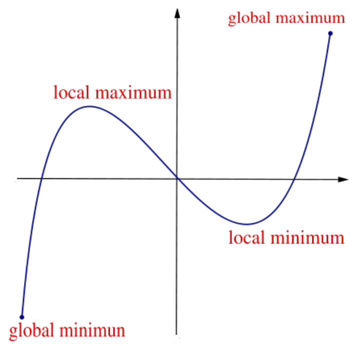
If f has a relative extremum at a point x_0 at which $f'(x_0)$ is defined, then $f'(x_0) = 0$.

GLOBAL EXTREMUM

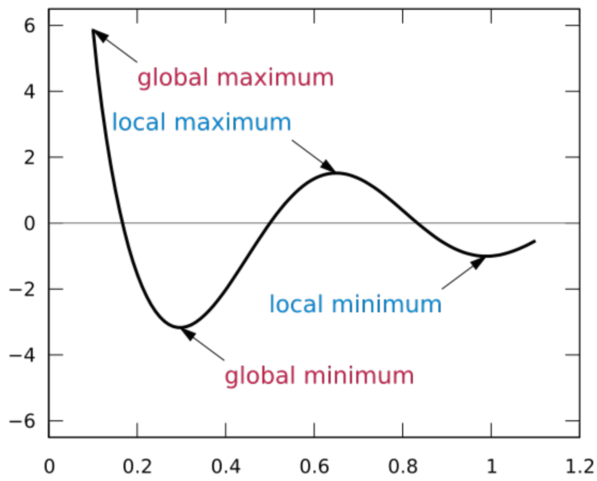
TERMINOLOGY: The terms local and global are synonymous with relative and absolute respectively. Also extremum is an inclusive term that includes both maximum and minimum: a local extremum is a local or relative maximum or minimum, and a global extremum is a global or absolute maximum or minimum.

We say that the function $f(x)$ has a global maximum at $x = x_0$ on the interval A, if $f(x_0) \geq f(x)$ for all $x \in A$. Similarly, the function $f(x)$ has a global minimum at $x = x_0$ on the interval A, if $f(x_0) \leq f(x)$ for all $x \in A$.

If $f(x)$ is a continuous function on a closed bounded interval $[a, b]$, then $f(x)$ will have a global maximum and a global minimum on $[a, b]$.



How can we find global extrema? Unfortunately, not every global extremum is also a local extremum.



Example

Consider the function $f(x) = (x-1)^2$, for $x \in [0,3]$. The only critical point is $x = 1$. And the first or second derivative test will imply that $x = 1$ is a local minimum. Looking at the graph we see that the right endpoint of the interval $[0,3]$ is the global maximum.

Remark

If $f(x)$ is differentiable on the interval A, then every global extremum is a local extremum or an endpoint extremum. This suggests the following strategy to find global extrema:

1. Find the critical points.
2. List the endpoints of the interval under consideration.
3. The global extrema of $f(x)$ can only occur at these points! Evaluate $f(x)$ at these points to check where the global maxima and minima are located.

Concavity and Points of Inflection

Definition – Critical Points

We will call the number c a first order critical number if $f'(c) = 0$ or $f'(c)$ does not exist and a second-order critical number if $f''(c) = 0$ or $f'(c)$ does not exist.

Definition – Concave Up and Concave Down

If the graph of f lies above all of its tangents on an interval A it is called concave up on A. If the graph of f lies below all of these tangents, it is called concave down on A.

Theorem

Let $f(x)$ be a differentiable function on an interval A.

- We will say that the graph of $f(x)$ is concave up on A if and only if $f'(x)$ is increasing on A.
- We will say that the graph of $f(x)$ is concave down on A if and only if $f'(x)$ is decreasing on A.

Some authors use the words concave for concave down and convex for concave up instead (we also use these words in Polish). Usually graphs have regions which are concave up and others which are concave down. Thus there are often points at which the graph changes from being concave up to concave down, or vice versa. These points are called inflection points.

Since the monotonicity behavior of a function is related to the sign of its derivative we get the following result:

Let $f(x)$ be a differentiable function on an interval A. Assume that $f'(x)$ is also differentiable on A.

- $f(x)$ is concave up on A if and only if $f''(x) > 0$ on A.
- $f(x)$ is concave down on A if and only if $f''(x) < 0$ on A.

It is clear from this result that if c is an inflection point then we must have: $f''(c) = 0$ or $f''(c)$ does not exist.

To determine concavity of a graph is similar to the method of finding increasing/decreasing intervals of a graph. Points of inflection are the same as critical points except they use the second derivative of f.

Finding Concavity

1. Locate the points of inflection and use these numbers to determine test intervals.
2. Determine the sign of $f''(x)$ at one value in each of the test intervals.
3. Use the Concavity Test Theorem to decide whether f is concave upward or downward on each interval.

Asymptotes

An asymptote of a real-valued function $y = f(x)$ is a curve which describes the behavior of f as either x or y tends to infinity. In other words, as one moves along the graph of $f(x)$ in some direction, the distance between it and the asymptote eventually becomes smaller than any distance that one may specify. An asymptote is, essentially, a line that a graph approaches, but does not intersect.

A function may have more than one asymptote.

If an asymptote is parallel with the y-axis, we call it a vertical asymptote. If an asymptote is parallel with the x-axis, we call it a horizontal asymptote. All other asymptotes are oblique asymptotes.

Asymptotes are formally defined using limits.

Vertical asymptotes

The line $x = a$ is a vertical asymptote of a function f if either of the following conditions is true:

$$\lim_{x\to a^-} f(x) = \pm\infty\ , \quad \lim_{x\to a^+} f(x) = \pm\infty\ .$$

Intuitively, if $x = \alpha$ is an asymptote of f, then, if we imagine x approaching α from one side, the value of $f(x)$ grows without bound – i.e., $f(x)$ becomes large (positively or negatively) – and, in fact, becomes larger than any finite value. For example the function $tan(x)$ has many vertical asymptotes.

In this case, we are finding the values of a that will make the function undefined - for example: for rational functions you should find the zeroes of the denominator.

Horizontal asymptotes

Suppose f is a function. Then the line $y = a$ is a horizontal asymptote for f if

$$\lim_{x\to -\infty} f(x) = a \quad \text{or} \quad \lim_{x\to +\infty} f(x) = a.$$

Note that if

$$\lim_{x\to -\infty} f(x) = a_1 \quad \text{and} \quad \lim_{x\to +\infty} f(x) = a_2$$

then the graph of f has two horizontal asymptotes: $y = a_1$ and $y = a_2$. An example of such a function is the arctangent function.

For rational function, when the degrees of the numerator and the denominator are the same, then the horizontal asymptote is found by dividing the leading terms, so the asymptote is given by:

$$y = \frac{numerator's\ leading\ coefficient}{denominator's\ leading\ coefficient}$$

and if the polynomial in the denominator has a bigger leading exponent than the polynomial in the numerator, then the graph trails along the x-axis at the far right and the far left of the graph, so the horizontal asymptote is $y = 0$.

Oblique asymptotes

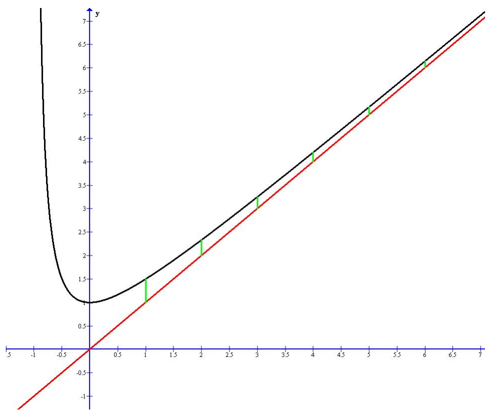
When an asymptote is not parallel to the x- or y-axis, it is called either an oblique asymptote or a slant asymptote. If $y = Ax + B$, is any non-vertical line, then the function f(x) is asymptotic to it if

$$\lim_{x\to-\infty}[f(x) - (Ax + B)] = 0 \quad \text{or} \quad \lim_{x\to+\infty}[f(x) - (Ax + B)] = 0 \ ,$$

where

$$A = \lim_{x\to\pm\infty} \tfrac{f(x)}{x} \ , \qquad B = \lim_{x\to\pm\infty}[f(x) - Ax] \ .$$

Computationally identifying an oblique asymptote can be more difficult than a horizontal or vertical asymptote.



Black: the graph of $f(x) = \dfrac{x^2 + x + 1}{x + 1}$.
Red: the asymptote $y = x$.
Green: difference between the graph and its asymptote for $x = 1, 2, 3, 4, 5, 6$.

A given rational function may or may not have a vertical asymptote (depending upon whether the denominator ever equals zero), but it will always have either a horizontal or else a slant asymptote.
Note, however, that the function will only have one of these two:

you will have either a horizontal asymptote or else a slant asymptote, but not both.

As soon as you see that you'll have one of them, don't bother looking for the other one!

Rational Functions and Asymptotes

A rational function is a function that can be written as the ratio of two polynomials (where the denominator isn't zero).

The equations of the vertical asymptotes can be found by finding the roots of the denominator.

The location of the horizontal asymptote is determined by looking at the degrees of the numerator (n) and denominator (m).

- If $n < m$, the x-axis, $y = 0$ is the horizontal asymptote.
- If $n = m$, then $y = a_n/b_m$ is the horizontal asymptote (that is, the ratio of the leading coefficients).
- If $n > m$, there is no horizontal asymptote. However, if $n = m + 1$, there is an oblique or slant asymptote.