THE IDEA - ANTIDERIVATIVES

The principles of integration were formulated by Isaac Newton and Gottfried Wilhelm Leibniz in the late seventeenth century. Through the fundamental theorem of calculus that they independently developed, integration is connected with differentiation, and the definite integral of a function can be easily computed once an antiderivative is known. Integrals and derivatives became the basic tools of calculus, with numerous applications in science and engineering.

The term "integral" may also refer to the notion of antiderivative, a function F whose derivative is the given function f. In this case it is called an indefinite integral.

$$F'(x) = f(x)$$

If a function has an integral, it is said to be integrable. The function for which the integral is calculated is called the integrand. The region over which a function is being integrated is called the domain of integration. If the integral does not have a domain of integration, it is considered indefinite.

DEFINITION

A function F is an antiderivative or an indefinite integral of the function f if the derivative $F' = f$. We use the notation

$$\int f(x)\,dx$$

to indicate that F is an indefinite integral of f. Using this notation, we have $F(x) = \displaystyle\int f(x)\,dx$ if and only if $F'(x) = f(x)$.

UNIQUENESS THEOREM

If F and G are antiderivatives of f on some interval A (i.e., $F'(x) = G'(x) = f(x)$ for all x in A) then there is a constant C such that $F(x) = G(x) + C$ for all x in A.

As a consequence of this theorem, we will usually add the constant C to an indefinite integral:

$$\int f(x)\,dx = F(x) + C\ .$$

There are many functions whose antiderivatives, even though they exist, cannot be expressed in terms of elementary functions (like polynomials, exponential functions, logarithms, trigonometric functions, inverse trigonometric functions and their combinations). Examples of these are $\int e^{-x^2} dx$, $\int \frac{\sin x}{x} dx$.

Inverse property of indefinite integrals

$$\left[\int f(x)dx\right]' = f(x)$$

$$\int f'(x)dx = f(x) + C$$

Antiderivative formulas

$$\int 0\, dx = C$$

$$\int x^p\, dx = \frac{x^{p+1}}{p+1} + C, \quad p \neq -1$$

$$\int \frac{1}{x}\, dx = \ln|x| + C, \quad x \neq 0$$

$$\int a^x\, dx = \frac{a^x}{\ln a} + C\ , \qquad \int e^x\, dx = e^x + C$$

$$\int \sin x\, dx = -\cos x + C\ , \qquad \int \cos x\, dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\, dx = \tan x + C, \quad x \neq \frac{\pi}{2} + k\pi$$

$$\int \frac{1}{\sin^2 x}\, dx = -\cot x + C, \quad x \neq k\pi$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C = -\arccos x + C, \quad x \in (-1,1)$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x + C = -\text{arccot}\ x + C$$

Antiderivative linearity rules

1. Sum Rule: $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$;
2. Difference Rule $\int (f(x) - g(x))\, dx = \int f(x)\, dx - \int g(x)\, dx$;
3. Constatnt Multiple Rule: $\int (a \cdot f(x))\, dx = a \cdot \int f(x)\, dx$, where $a \in \mathbf{R}$.

Examples

Example 1. $\int (2x^2 + 3x - \frac{1}{x} + \frac{3}{1+x^2})dx = \int 2x^2 dx + \int 3x dx - \int \frac{dx}{x} + 3\int \frac{dx}{1+x^2} =$
$= 2 \cdot \frac{x^3}{3} + 3 \cdot \frac{x^2}{2} - \ln|x| + 3\arctan x + C.$

Example 2. $\int \frac{\sin^2 x - \cos^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{dx}{\cos^2 x} - \int \frac{dx}{\sin^2 x} = \tan x + \cot x + C.$

Example 3. $\int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1-1}{1+x^2} dx = \int 1 dx - \int \frac{1}{1+x^2} = x + \text{arccot}\ x + C.$

ANTIDERIVATIVE FORMULAS FOR FUNCTIONS AND THEIR DERIVATIVES

$$\int (f(x))^n \cdot f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + C$$
$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$
$$\int \frac{f'(x)}{(f(x))^2}dx = \frac{1}{f(x)} + C$$
$$\int e^{f(x)} \cdot f'(x) = e^{f(x)} + C$$
$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C$$

EXAMPLES

Example 4. $\int \sin^5 x \cdot \cos x dx = [template: \ (f(x))^n \cdot f'(x)] = \int \sin^6 x \cdot (\sin x)' dx = \frac{\sin^6 x}{6} + C.$

Example 5. $\int \frac{e^x}{e^x+1}dx = [template: \ \frac{f'(x)}{f(x)}] = \int \frac{(e^x+1)'}{e^x+1}dx = \ln|e^x| + C = \ln e^x + C = x + C.$

Example 6. $\int \frac{e^{\arctan x}}{1+x^2}dx = [template: \ e^{f(x)} \cdot f'(x)] = e^{\arctan x} + C.$

Example 7. $\int \frac{\sin x dx}{\sqrt{2-\cos x}} = [template: \ \frac{f'(x)}{\sqrt{f(x)}}] = \int \frac{(2-\cos x)'}{\sqrt{2-\cos x}}dx = 2\sqrt{2-\cos x} + C.$

INTEGRATION BY PARTS

If u and v have continuous derivatives, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

EXAMPLES

Example 8. $\int \ln x dx = \int \ln x dx = \int (\ln x) \cdot 1 dx = \left| \begin{array}{ll} f(x) = \ln x & f'(x) = \frac{1}{x} \\ g'(x) = 1 & g(x) = x \end{array} \right| = x\ln x - \int \frac{x}{x}dx = x\ln x - \int 1dx = x\ln x - x + C = x(\ln x - 1) + C.$

Example 9. $\int \cos^2 x dx = \int \cos x \cos x dx = \left| \begin{array}{ll} f(x) = \cos x & f'(x) = -\sin x \\ g'(x) = \cos x & g(x) = \sin x \end{array} \right| = \sin x \cos x + \int \sin^2 x dx = \sin x\cos x + \int (1-\cos^2 x)dx = \sin x \cos x + x + C - \int \cos^2 x dx.$

We obtained:

$$\int \cos^2 x dx = \cdots = \sin x \cos x + x + C - \int \cos^2 x dx$$

therefore we need to simplify this equation:

$$\begin{aligned} \int \cos^2 x dx &= \sin x \cos x + x + C - \int \cos^2 x dx \\ 2 \cdot \int \cos^2 x dx &= \sin x \cos x + x + C \\ \int \cos^2 x dx &= \frac{\sin x \cos x + x}{2} + C \end{aligned}$$

We remember that $\sin 2x = 2\sin x\cos x$, so finally:

$$\int \cos^2 x dx = \frac{\frac{\sin 2x}{2} + x}{2} + C = \frac{\sin 2x + 2x}{4} + C.$$

INTEGRATION BY SUBSTITUTION

If $u = g(x)$ is a diferentiable function whose range is an interval A and f is continuous in A, then

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt = F(g(t)) + C\ .$$

EXAMPLES

Example 10. $\int \sin^3 x \cos x dx = \left| \begin{array}{c} t = \sin x \\ dt = \sin'(x) = \cos x dx \end{array} \right| = \int t^3 dt = \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C.$

Example 11. $\int \frac{x dx}{\sqrt{1-x^4}} = \left| \begin{array}{c} t = x^2 \\ dt = (x^2)' = 2x dx \\ \frac{dt}{2} = x dx \end{array} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\arcsin t + C = \frac{1}{2}\arcsin x^2 + C.$

Example 12. $\int \frac{dx}{e^x+e^{-x}} = \int \frac{dx}{e^{-x}(e^{2x}+1)} = \int \frac{e^x dx}{1+e^{2x}} = \left| \begin{array}{c} t = e^x \\ dt = (e^x)' = e^x dx \end{array} \right| = \int \frac{dt}{1+t^2} = \arctan t + C = \arctan e^x + C.$

INTEGRALS OF THE FORM: $\int \frac{dx}{x^2+\alpha x+\beta}$, $\Delta < 0$ - AN EXAMPLE

Example 13. $\int \frac{dx}{x^2+3x+4} = \int \frac{dx}{x^2+\underbrace{2\cdot\frac{3}{2}}_{=3}x+\underbrace{\frac{9}{4}+\frac{7}{4}}_{=4}} = \int \frac{dx}{(x+\frac{3}{2})^2+\frac{7}{4}} = \int \frac{dx}{\underbrace{\frac{7}{4}\cdot\frac{4}{7}}_{=1}(x+\frac{3}{2})^2+\frac{7}{4}} = \frac{1}{\frac{7}{4}}\int \frac{dx}{\frac{4}{7}(x+\frac{3}{2})^2+1} =$

$\frac{4}{7}\int \frac{dx}{(\frac{2}{\sqrt{7}})^2(x+\frac{3}{2})^2+1} = \frac{4}{7}\int \frac{dx}{(\frac{2x}{\sqrt{7}}+\frac{3}{\sqrt{7}})^2+1} = \left| \begin{array}{c} t = \frac{2x}{\sqrt{7}} + \frac{3}{\sqrt{7}} \\ dt = \frac{2}{\sqrt{7}}dx \\ \frac{\sqrt{7}}{2}dt = dx \end{array} \right| = \frac{4}{7}\int \frac{\frac{\sqrt{7}}{2}dt}{t^2+1} = \frac{2}{\sqrt{7}}\int \frac{dt}{t^2+1} =$

$= \frac{2}{\sqrt{7}}\arctan t + C = \frac{2}{\sqrt{7}}\arctan\left(\frac{2x}{\sqrt{7}} + \frac{3}{\sqrt{7}}\right) + C.$

In general, $\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \cdot \arctan(\frac{2ax+b}{\sqrt{4ac-b^2}})$, for $b^2 - 4ac < 0$.

If $b^2 - 4ac > 0$, then one has to perform integration by partial fractions.

INTEGRATION BY PARTIAL FRACTIONS

In general, we have three kinds of partial fractions:

$$\int \frac{Adx}{(x+a)} = A \cdot \ln|x+a| + C, \qquad A, C \in \mathbf{R}$$
$$\int \frac{Adx}{(x+a)^n} = \frac{-A}{(n-1)(x+a)^{n-1}} + C, \qquad A, C \in \mathbf{R},\ n \geq 2$$
$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-2)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)} \int \frac{dx}{(1+x^2)^{n-1}}, \qquad n \geq 2$$

EXAMPLES

Example 14. $\int \frac{6dx}{x^2-1} = \int \frac{6dx}{(x-1)(x+1)} = \int (\frac{A}{x-1} + \frac{B}{x+1})dx = \dots$

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} = \frac{x(A+B)+A-B}{x^2-1} = \frac{0 \cdot x+6}{x^2-1}$$
$$\begin{cases} A+B=0 \\ A-B=6 \end{cases}$$
$$Solution: A=3, B=-3.$$

$\dots = \int (\frac{3}{x-1} - \frac{3}{x+1})dx = \int \frac{3dx}{x-1} - \int \frac{3dx}{x+1} = 3\ln|x-1| - 3\ln|x+1| + C = 3\ln|\frac{x-1}{x+1}| + C.$

Example 15. $\int \frac{x-1}{x^2+x}dx = \int \frac{x-1}{x(x+1)} = \int (\frac{A}{x} + \frac{B}{x+1})dx = \dots$

$$\frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{x(A+B)+A}{x^2+x} = \frac{x-1}{x^2+x}$$
$$\begin{cases} A+B=1 \\ A=-1 \end{cases}$$
$$Solution: A=-1, B=2.$$

$\dots = \int (\frac{-1}{x} + \frac{2}{x+1})dx = \int \frac{-dx}{x} + \int \frac{2dx}{x+1} = -\ln|x| + 2\ln|x+1| + C = -\ln|x| + \ln|x+1|^2 + C =$
$= \ln\frac{(x+1)^2}{|x|} + C.$

Example 16. $\int \frac{x^2dx}{(x+3)(x^2-3x+3)} = [\Delta_{(x^2-3x+3)} < 0] = \int (\frac{A}{x+3} + \frac{Bx+C}{x^2-3x+3})dx = \dots$

$$\frac{A}{x+3} + \frac{Bx+C}{x^2-3x+3} = \frac{A(x^2-3x+3)+(Bx+C)(x+3)}{(x+3)(x^2-3x+3)} = \frac{x^2(A+B)+x(-3A+3B+C)+(3A+3C)}{(x+3)(x^2-3x+3)} = \frac{x^2+0 \cdot x+0}{(x+3)(x^2-3x+3)}$$
$$\begin{cases} A+B=1 \\ -3A+3B+C=0 \\ 3A+3C=0 \end{cases}$$
$$Solution: A=\frac{3}{7}, B=\frac{4}{7}, C=-\frac{3}{7}.$$

$$\dots = \int \frac{\frac{3}{7}dx}{x+3} + \frac{1}{7} \underbrace{\int \frac{4x-3}{x^2-3x+3}dx}_{I} = \dots$$

$$I = \int \frac{4x-3}{x^2-3x+3}dx = \int \frac{4x-6+3}{x^2-3x+3}dx = \int \frac{2(x^2-3x+3)'}{x^2-3x+3}dx + 3\int \frac{dx}{x^2-3x+3} =$$
$$2\ln|x^2-3x+3| + \frac{2}{\sqrt{3}}\arctan(\frac{2x-3}{\sqrt{3}}) + C$$

$\dots = \frac{3}{7}\ln|x+3| + \frac{2}{7}\ln|x^2-3x+3| + \frac{2\sqrt{3}}{7}\arctan(\frac{2x-3}{\sqrt{3}}) + C.$

HYPERBOLIC FUNCTIONS

The *hyperbolic sine* and *hyperbolic cosine* are defined as:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Instead of *sinh* and *cosh* we will usually write *sh* and *ch*.

Hyperbolics functions hold the following properties:

$$\begin{aligned} sh'(x) &= ch(x) \\ ch'(x) &= sh(x) \\ 1 &= ch^2(x) - sh^2(x) \\ e^x &= sh(x) + ch(x) \\ x &= \ln(sh(x) + ch(x)) \end{aligned}$$

$$\begin{aligned} \int sh^2(x)dx &= \frac{sh(x)ch(x) - x}{2} + C \\ \int ch^2(x)dx &= \frac{sh(x)ch(x) + x}{2} + C \end{aligned}$$

We will use hyperbolic functions to integrate algebraic functions.

INTEGRATION OF ALGEBRAIC FUNCTIONS

We will distinguish between two types of algebraic functions:

Type 1. If the function contains $\sqrt{x^2 - a^2}$, then:

$$\begin{aligned} x &= a \cdot ch(t) \\ dx &= a \cdot sh(t)dt \\ \sqrt{x^2 - a^2} &= a \cdot sh(t) \end{aligned}$$

Type 2. If the function contains $\sqrt{x^2+a^2}$, then:

$$\begin{aligned} x &= a\cdot sh(t) \\ dx &= a\cdot ch(t)dt \\ \sqrt{x^2+a^2} &= a\cdot ch(t) \end{aligned}$$

Example 17. $\int \sqrt{x^2-4}dx = \left| \begin{array}{c} a=2 \\ x=2ch(t) \\ dx=2sh(t)dt \\ \sqrt{x^2-4}=2sh(t) \end{array} \right| = \int 4sh^2(t)dt = 4\cdot\frac{sh(t)ch(t)-t}{2}+C =$
$2(\frac{\sqrt{x^2-4}}{2}\cdot\frac{x}{2}-\ln(sh(t)+ch(t)))+C = 2(\frac{x\sqrt{x^2-4}}{4}-\ln(\frac{\sqrt{x^2-4}}{2}+\frac{x}{2}))+C.$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Remember these important formulas:

$$\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}\Big(\cos(\alpha-\beta)-\cos(\alpha+\beta)\Big) \\ \cos\alpha\cos\beta &= \frac{1}{2}\Big(\cos(\alpha-\beta)+\cos(\alpha+\beta)\Big) \\ \sin\alpha\cos\beta &= \frac{1}{2}\Big(\sin(\alpha-\beta)+\sin(\alpha+\beta)\Big) \end{aligned}$$

Example 18. $\int \cos 5x\cos(3x+\frac{\pi}{2})dx = \int\frac{1}{2}(\cos(5x-3x-\frac{\pi}{2})+\cos(5x+3x+\frac{\pi}{2}))dx =$
$= \frac{1}{2}\int\cos(2x-\frac{\pi}{2})dx+\frac{1}{2}\int\cos(8x+\frac{\pi}{2})dx = \left| \begin{array}{c} t=2x-\frac{\pi}{2} \\ dt=2dx \\ \frac{dt}{2}=dx \end{array} \right| \text{and} \left| \begin{array}{c} s=8x+\frac{\pi}{2} \\ ds=8dx \\ \frac{ds}{8}=dx \end{array} \right| =$
$= \frac{1}{2}\int\cos t\frac{dt}{2}+\frac{1}{2}\int\cos s\frac{ds}{8} = \frac{1}{4}\int\cos t dt+\frac{1}{16}\int\cos s ds = \frac{1}{4}\sin t+\frac{1}{16}\sin s+C =$
$= \frac{1}{4}\sin(2x-\frac{\pi}{2})+\frac{1}{16}\sin(8x+\frac{\pi}{2})+C.$

If a function is made up of trigonometric functions but is not similar to above formulas, then we need to use substitution:

$$\begin{aligned} t &= \tan\frac{x}{2} \\ x &= 2\arctan t \\ dx &= \frac{2dt}{1+t^2} \\ \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \end{aligned}$$

Example 19. $\int\frac{dx}{3\sin x-4\cos x} = \int\frac{1}{\frac{6t}{1+t^2}-4\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = \int\frac{2dt}{4t^2+6t-4} = \int\frac{dt}{2t^2+3t-2} = \cdots = \int\frac{dt}{2(t-\frac{-3-\sqrt{41}}{4})(t-\frac{-3+\sqrt{41}}{4})} =$
finish by yourself using the method of partial fractions :).