DEFINITION

An infinite sequence (or sequence) of numbers is a function whose domain is the set of positive integers. The number a_n is called the nth term of the sequence, or the term with index n.

The graph of the sequence (a_n) consists of those points in the xy-plane for which $x = n$ and $y = a_n$ for $n \in \mathbf{N}$. The points must not be connected with a line, which is an often-made mistake.

REMARK

Usually, n_1 is 1, and the domain is the set of all natural numbers. But sometimes we want to start our sequence elsewhere.

MONOTONICITY

Sequence (a_n) is:

- increasing – if $a_{n+1} > a_n$ for every $n \in \mathbf{N}$,
- nondecreasing – if $a_{n+1} \geq a_n$ for every $n \in \mathbf{N}$,
- nonincreasing – if $a_{n+1} \leq a_n$ for every $n \in \mathbf{N}$,
- decreasing – if $a_{n+1} < a_n$ for every $n \in \mathbf{N}$.

To check monotonicity, it is enough to check the sign of $a_{n+1} - a_n$. If:

- $a_{n+1} - a_n > 0$ for every $n \in \mathbf{N}$, then the sequence is increasing,
- $a_{n+1} - a_n \geq 0$ for every $n \in \mathbf{N}$, then the sequence is nondecreasing,
- $a_{n+1} - a_n \leq 0$ for every $n \in \mathbf{N}$, then the sequence is nonincreasing,
- $a_{n+1} - a_n < 0$ for every $n \in \mathbf{N}$, then the sequence is decreasing.

If all terms of a sequence (a_n) are positive, then we can also check the value of $\frac{a_{n+1}}{a_n}$:

- $\frac{a_{n+1}}{a_n} > 1$ for every $n \in \mathbf{N}$, then the sequence is increasing,
- $\frac{a_{n+1}}{a_n} \geq 1$ for every $n \in \mathbf{N}$, then the sequence is nondecreasing,
- $\frac{a_{n+1}}{a_n} \leq 1$ for every $n \in \mathbf{N}$, then the sequence is nonincreasing,
- $\frac{a_{n+1}}{a_n} < 1$ for every $n \in \mathbf{N}$, then the sequence is decreasing.

Example 1. Let us consider sequence $a_n = -3^n + 2n + 1$. To establish monotonicity, we need to check the sign of $a_{n+1} - a_n$.

$$a_{n+1} - a_n = -3^{n+1} + 2(n+2) + 1 - (-3^n + 2n + 1) = -2 \cdot 3^n + 2 < 0 \text{ for every } n \in \mathbf{N}.$$

Thus, sequence a_n is decreasing.

Example 2. Now let us consider sequence $b_n = \frac{4^n}{(n+4)!}$. Every term of this sequence is positive, so we will check the value of $\frac{b_{n+1}}{b_n}$.

$$\frac{b_{n+1}}{b_n} = \frac{4^{n+1}}{(n+5)!} \cdot \frac{(n+4)!}{4^n} = \frac{4 \cdot (n+4)!}{(n+4)!(n+5)} = \frac{4}{n+5} < 1 \text{ for every } n \in \mathbf{N}.$$

This sequence is also decreasing.

Example 3. Let $c_n = (-1)^n \cdot n$. The first few terms of this sequence are:

-1, 2, -3, 4, -5, 6, ...

We can easily see that this sequence is nonmonotonic.

DEFINITION

Sequence (a_n) is bounded from below if there exists number $m \in \mathbf{R}$ such that $a_n \geq m$ for every $n \in \mathbf{N}$.

Sequence (a_n) is bounded from above if there exists number $M \in \mathbf{R}$ such that $a_n \leq M$ for every $n \in \mathbf{N}$.

Sequence (a_n) is bounded if it is bounded from below and from above.

Example 4. Sequence $a_n = 1 + \frac{2}{n}$ is bounded from above by 3 and from below by 1.

Example 5. Sequence $a_n = n^n$ is bounded only from below by 1.

Example 6. Sequence $a_n = (-1)^n \cdot n$ is not bounded.

DEFINITION

Sequence (a_n) is an arithmetic progression if $a_{n+1} - a_n = r$ for every $n \in \mathbf{N}$. Number r is called the common difference of the progression.

The sum of the first n terms of the arithmetic progression is equal to:

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{a_1 + a_n}{2} \cdot n$$

Example 7. We know that $a_1 = 6$ and $a_8 = 34$. Find $a_2, \ldots, a_7$ so that (a_n) becomes an arithmetic progression.

Solution: We know that $a_1 = 6$ and $a_8 = a_1 + 7r = 34$. Therefore $r = 4$. The missing numbers are: $a_2 = a_1 + r = 10, a_3 = a_1 + 2r = 14, a_4 = 18, a_5 = 22, a_6 = 26, a_7 = 30$.

DEFINITION

Sequence (a_n) is a geometric progression if $\frac{a_{n+1}}{a_n} = q$ for every $n \in \mathbf{N}$. Number q is called the common ratio of the progression.

The sum of the first n terms of the geometric progression is equal to:

$$S_n = a_1 + a_2 + \cdots + a_n = \begin{cases} n \cdot a_1 & q = 1 \\ \frac{a_1(1-q^n)}{1-q} \cdot n & q \neq 1 \end{cases}$$

Example 8. Find the first term of a geometric progression in which the common ratio is 2 and the sum of eight first terms is 765.

Solution: We know that $q = 2$ and $S_8 = 765$. So, $a_1 \frac{1-2^8}{1-2} = 765$. Hence $a_1 = \frac{765}{255} = 3$.

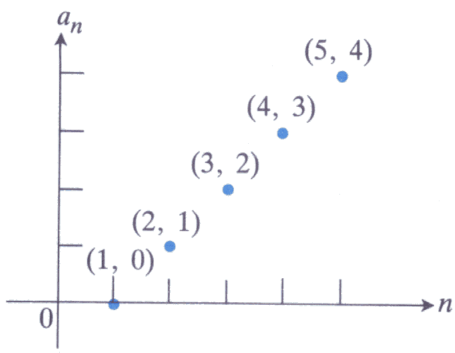
DEFINITION

The sequence (a_n) converges to the number g if to every positive number ϵ there corresponds an index $n_0 \in \mathbf{N}$ such that $(n > n_0) \implies (|a_n - g| < \epsilon)$ for all $n \in \mathbf{N}$.

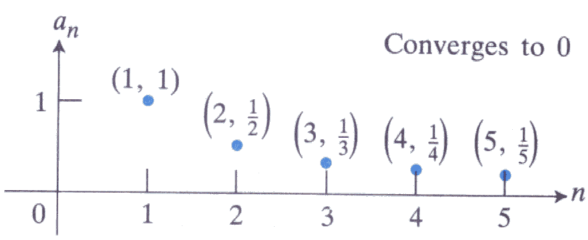
If no such limit exists, we say that (a_n) diverges.

We call g the limit of the sequence. It may happen that $g = \pm\infty$.

EXAMPLES

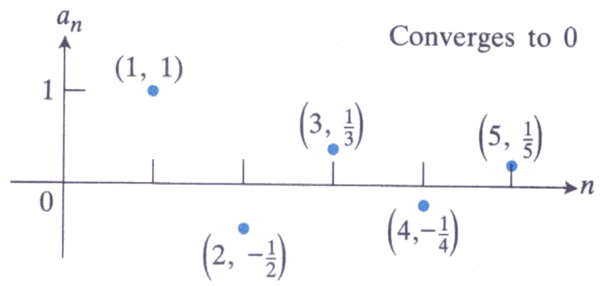


$a_n = n - 1$

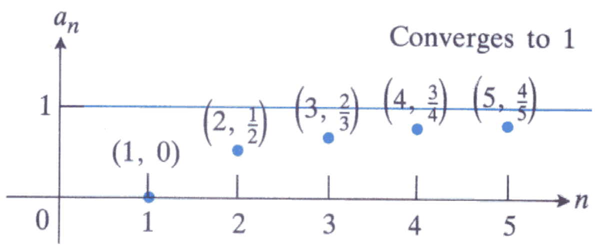


$a_n = \frac{1}{n}$

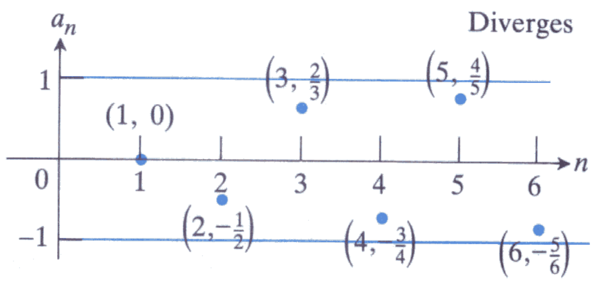
$a_n = \frac{(-1)^{n+1}}{n}$



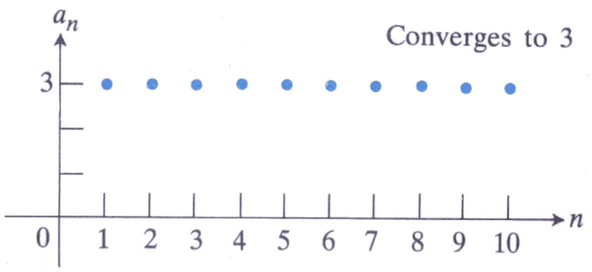
$a_n = \frac{n-1}{n}$



$a_n = (-1)^n \cdot \frac{n-1}{n}$



$a_n = 3$



Limit Theorems

If sequences (a_n) and (b_n) converge (both limits exist and are finite), then

- Sum/Difference Rule: $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$,
- Constant Multiple Rule: $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$, where $c \in R$,
- Product Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$,
- Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$,
- $\lim_{n\to\infty}(a_n)^p = (\lim_{n\to\infty} a_n)^p$, where $p \in Z \setminus \{0\}$,

- $\lim\limits_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n}$, where $k \in N$.

REMARK

If the sequence (a_n) diverges, and if c is any number different then 0, then the sequence (ca_n) also diverges.

THEOREM

If $\lim\limits_{n\to\infty} a_n = g$, and if f is a function that is continuous at g and definied at all the a_n's, then $\lim\limits_{n\to\infty} f(a_n) = f(g)$.

THEOREM

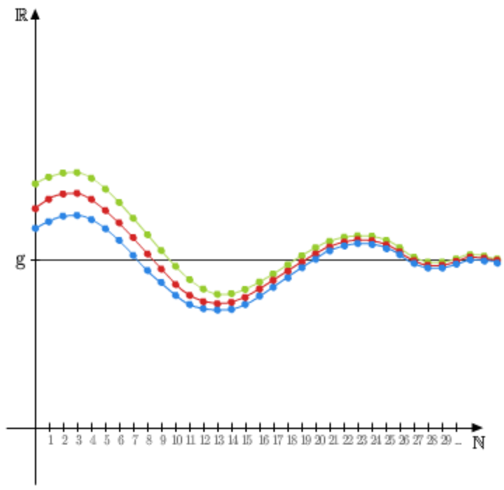
If (a_n) converges, then (a_n) is bounded. If (a_n) is unbounded, then (a_n) diverges.

LIMITS THAT ARISE FREQUENTLY

- $\lim\limits_{n\to\infty} q^n = \begin{cases} \text{does not exist} & q \leq -1 \\ 0 & |q| < 1 \\ 1 & q = 1 \\ \infty & q > 1 \end{cases}$.
- If $a > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

THE SANDWICH THEOREM FOR SEQUENCES

If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = g$, then $\lim\limits_{n\to\infty} b_n = g$.



Theorem of two sequences

If $a_n \le b_n$ for every $n \ge n_0 \in \mathbf{N}$ and $\lim\limits_{n\to\infty} a_n = \infty$, then $\lim\limits_{n\to\infty} b_n = \infty$ as well.

Theorem

Sequence $\left(1+\dfrac{1}{n}\right)^n$ converges to e ($\approx 2,7182818$).

Remark

If $\lim\limits_{n\to\infty} a_n = \infty$, then $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{a_n}\right)^{a_n} = e$.

Examples

Example 9. $\lim\limits_{n\to\infty} \frac{-4n^2+n-9}{3n^6+2n-1} = \lim\limits_{n\to\infty} \frac{-\frac{4}{n^4}+\frac{1}{n^5}-\frac{9}{n^6}}{3+\frac{2}{n^5}-\frac{1}{n^6}} = \frac{0}{3} = 0.$

Example 10. $\lim\limits_{n\to\infty} \frac{5\cdot 9^n-5^{n+1}+3}{-3^{2n}+2^{3n+1}-1} = \lim\limits_{n\to\infty} \frac{5\cdot 9^n-5\cdot 5^n+3}{-9^n+2\cdot 8^n-1} = \lim\limits_{n\to\infty} \frac{5-5\cdot(\frac{5}{9})^n+\frac{3}{9^n}}{-1+2\cdot(\frac{8}{9})^n-\frac{1}{9^n}} = -5.$

Example 11. $\lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+\sqrt{n+\sqrt{n}}}} = \lim\limits_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n+\sqrt{n+\sqrt{n}}}}{\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{\frac{n}{n}+\sqrt{\frac{n}{n^2}+\sqrt{\frac{n}{n^4}}}}} =$
$= \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\sqrt{\frac{1}{n}+\sqrt{\frac{1}{n^3}}}}} = 1.$

Example 12. $\lim\limits_{n\to\infty} \sqrt[n]{e^n+\pi^n+8^n}$=?.

$$\text{We know that } 8^n \le e^n+\pi^n+8^n \le 8^n+8^n+8^n = 3\cdot 8^n.$$
$$\text{So: } \sqrt{8^n} \le \sqrt{e^n+\pi^n+8^n} \le \sqrt{3\cdot 8^n}.$$
$$\text{We also know that } \lim_{n\to\infty} \sqrt[n]{8^n} = 8 \text{ and } \lim_{n\to\infty} \sqrt[n]{3\cdot 8^n} = \lim_{n\to\infty} 8\cdot\sqrt[n]{3} = 8.$$
$$\text{Thus } \lim_{n\to\infty} \sqrt[n]{e^n+\pi^n+8^n} = 8.$$

Example 13. $\lim\limits_{n\to\infty} (4-\arctan n)^n$ =?.

$$\text{We know that } -\tfrac{\pi}{2} < \arctan n < \tfrac{\pi}{2}. \text{ So, it is true that:}$$
$$4-\arctan n > 4-\tfrac{\pi}{2} > 2 \text{ and } (4-\arctan n)^n > 2^n.$$
$$\text{Since } \lim_{n\to\infty} 2^n = \infty, \text{ then by the theorem of two sequences we have:}$$
$$\lim_{n\to\infty} (4-\arctan n)^n = \infty.$$

Example 14. $\lim\limits_{n\to\infty} (\frac{n+5}{n})^{3n} = \lim\limits_{n\to\infty} (1+\frac{5}{n})^{3n} = \lim\limits_{n\to\infty} [(1+\frac{1}{\frac{n}{5}})^{\frac{n}{5}}]^{15} = e^{15}.$

Example 15. $\lim\limits_{n\to\infty} (\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}) = \lim\limits_{n\to\infty} \frac{1+2+\cdots+n-1}{n} = \lim\limits_{n\to\infty} \frac{\frac{(1+(n-1))(n-1)}{2}}{n} =$
$= \lim\limits_{n\to\infty} \frac{n-1}{2} = \infty.$

The sum of an infinite geometric sequence

If a_n is a geometric sequence with the common ratio $|q| < 1$ then $S = a_1 + a_2 + \dots = \frac{a_1}{1-q}$.

Example 16. Calculate $7 + 2.1 + 0.63 + \dots$.

Solution. It is easy to see that $q = 0.3 < 1$, so $S = \frac{7}{1-0.3} = 10$.

Example 17. Express 1.24(36) as a common fraction.

Solution $1.24(36) = 1.2436363636\cdots = 1.24 + (0.0036 + 0.000036 + \dots) = 1.24 + \frac{0.36}{1-0.01} = 1.24 + \frac{36}{9900} = \frac{342}{275}$.