Double Integrals - Techniques and Examples

Iterated integrals on a rectangle

If function f is continuous on an integral $[a,b]\times[c,d]$, then:

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\ dxdy = \int_a^b \Big(\int_c^d f(x,y)\ dy\Big)\ dx = \int_c^d \Big(\int_a^b f(x,y)\ dx\Big)\ dy.$$

Notation

Instead of $\int_a^b \big(\int_c^d f(x,y)\ dy\big)\ dx$ we may also write $\int_a^b dx \int_c^d f(x,y)\ dy$.

Instead of $\int_c^d \big(\int_a^b f(x,y)\ dx\big)\ dy$ we may also write $\int_c^d dy \int_a^b f(x,y)\ dx$.

Example 1. Calculate $\iint\limits_R \frac{x}{y^2}\ dxdy$, where $R=[1,2]\times[4,6]$.

Solution: $\iint\limits_R \frac{x}{y^2}\ dxdy = \iint\limits_{[1,2]\times[4,6]} \frac{x}{y^2}\ dxdy = \int_1^2 \big(\int_4^6 \frac{x}{y^2}\ dy\big)\ dx = \int_1^2 \big([-\frac{x}{y}]_{y=4}^{y=6}\big)\ dx = \int_1^2 \big(\frac{x}{4}-\frac{x}{6}\big)\ dx = \int_1^2 \frac{x}{12}\ dx = [\frac{x^2}{24}]_{x=1}^{x=2} = \frac{4-1}{24} = \frac{3}{24} = \frac{1}{8}$.

A double integral of a function with separable variables

If function f is of form $f(x,y)=g(x)\cdot h(y)$ and g is continuous in $[a,b]$ and h is continuous in $[c,d]$, then:

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\ dxdy = \Big(\int_a^b g(x)dx\Big)\cdot\Big(\int_c^d h(y)\ dy\Big).$$

Example 2. Calculate $\iint\limits_R \frac{x}{y^2}\ dxdy$, where $R=[1,2]\times[4,6]$, separating variables.

Solution: $\iint\limits_R \frac{x}{y^2}\ dxdy = \iint\limits_R x\cdot\frac{1}{y^2}\ dxdy = \Big(\int_1^2 x\ dx\Big)\cdot\Big(\int_4^6 \frac{dy}{y^2}\Big) = \big(\ [\frac{x^2}{2}]_{x=1}^{x=2}\ \big)\cdot\big(\ [-\frac{1}{y}]_{y=4}^{y=6}\ \big) = \big(\frac{4-1}{2}\big)\cdot\big(-\frac{1}{6}+\frac{1}{4}\big) = \frac{3}{2}\cdot\frac{-2+3}{12} = \frac{3}{24} = \frac{1}{8}$.

A double integral over a simple region

If f is a continuous function on the vertically simple region

$$D=\{(x,y): a\le x\le b,\ g(x)\le y\le h(x)\},$$

then

$$\iint_D f(x,y)\,dP = \int_a^b \Big(\int_{g(x)}^{h(x)} f(x,y)\,dy \Big)\,dx.$$

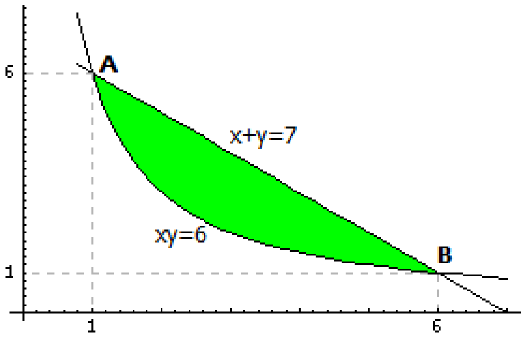
If f is a continuous function on the horizontally simple region

$$D = \{(x,y) : c \le y \le d,\ p(y) \le x \le q(y)\},$$

then

$$\iint_D f(x,y)\,dP = \int_c^d \Big(\int_{p(y)}^{q(y)} f(x,y)\,dx \Big)\,dy.$$

Example 3. Evaluate $\iint_D (x+y)\,dxdy$ over a region bounded by curves $xy = 6$ and $x + y = 7$. Sketch a diagram of the region.

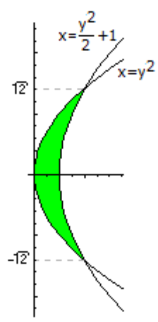


Solution: From the system of equations of $xy = 6$ and $x + y = 7$ (or: $y = \frac{6}{x}$, $y = 7 - x$) we obtain two intersection points: $A = (1, 6)$ and $B = (6, 1)$. Region D is vertically simple, so:

$$\begin{aligned}
\iint_D (x+y)\,dxdy &= \int_1^6 \Big(\int_{y=\frac{6}{x}}^{y=7-x} (x+y)\,dy \Big)\,dx = \int_1^6 \Big([xy + \frac{y^2}{2}]_{y=\frac{6}{x}}^{y=7-x} \Big)\,dx \\
&= \int_1^6 \Big(x(7-x) + \frac{(7-x)^2}{2} - x \cdot \frac{6}{x} - \frac{36}{2x^2} \Big)\,dx \\
&= \int_1^6 (-\frac{x^2}{2} - \frac{18}{x^2} + \frac{37}{2})\,dx = [-\frac{x^3}{6} + \frac{18}{x} + \frac{37x}{2}]_1^6 = \frac{125}{3}.
\end{aligned}$$

Example 4. Evaluate $\iint_D (x - y)\,dxdy$ over a region bounded by curves $x = y^2$ and $x = \frac{y^2}{2} + 1$. Sketch a diagram of the region.

Solution: From the system of equations of $x = y^2$ and $x = \frac{y^2}{2} + 1$ we obtain two intersection points: $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. Region D is horizontally simple, so:



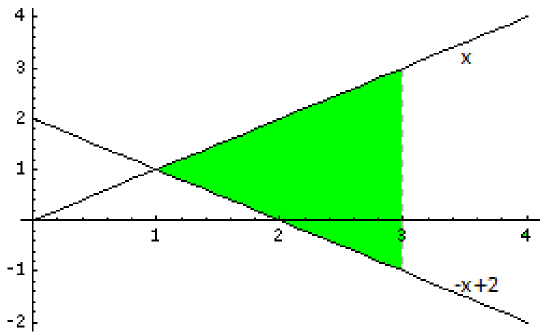
$$\iint_D (x-y)\,dxdy = \int_{-\sqrt{2}}^{\sqrt{2}} \Big(\int_{y^2}^{\frac{y^2}{2}+1} (x-y)\,dx \Big)\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \Big([\frac{x^2}{2} - xy]_{x=y^2}^{x=\frac{y^2}{2}+1} \Big)\,dy =$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \Big(\frac{(\frac{y^2}{2}+1)^2}{2} - (\frac{y^2}{2}+1)y - \frac{y^4}{2} + y^3 \Big)\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \Big(-\frac{3y^4}{8} + \frac{y^3}{2} + \frac{y^2}{2} - y + \frac{1}{2} \Big)\, dy =$$

$$= \Big[-\frac{3y^5}{40} + \frac{y^4}{8} + \frac{y^3}{6} - \frac{y^2}{2} + \frac{y}{2} \Big]_{-\sqrt{2}}^{\sqrt{2}} = \frac{16\sqrt{2}}{15}.$$

Iterated integrals in a reversed order

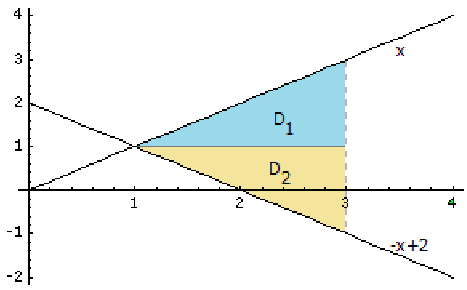
Example 5. Sketch the region over which the integration $\int_1^3 \int_{-x+2}^{x} (2x+1)\, dydx$ takes place and write an equivalent integral with the order of integration reversed. Evaluate both integrals.



Solution: First let us evaluate:

$$\int_1^3 \int_{-x+2}^{x} (2x+1)\, dydx = \int_1^3 \Big([y(2x+1)]_{-x+2}^{x} \Big)\, dx = \int_1^3 \Big(x(2x+1) - (-x+2)(2x+1) \Big)\, dx$$

$$= \int_1^3 (-2 - 2x + 4x^2)\, dx = \Big[-2x - x^2 + \frac{4x^3}{3}\Big]_1^3 = \frac{68}{3}.$$



To reverse the order of integration, we need to divide the region into two parts that are horizontally simple. Now:

$$\int_1^3 \int_{-x+2}^{x} (2x+1)\, dydx = \int_1^3 \int_y^3 (2x+1)\, dxdy + \int_{-1}^1 \int_{-y+2}^{3} (2x+1)\, dxdy$$

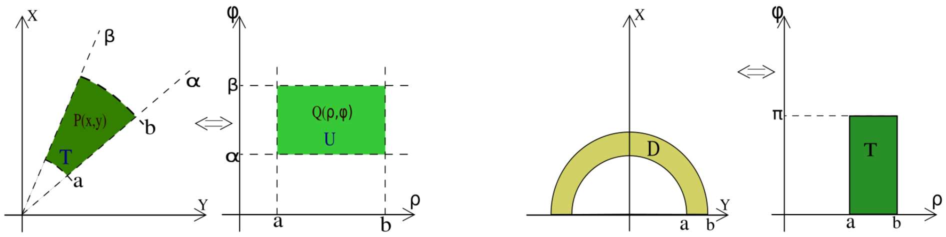
$$= \int_1^3 [x+x^2]_y^3\, dy + \int_{-1}^1 [x+x^2]_{-y+2}^3\, dy = \int_1^3 (12 - y - y^2)\, dy + \int_{-1}^1 (6 + 5y - y^2)\, dy$$

$$= \Big[12y - \frac{y^2}{2} - \frac{y^3}{3}\Big]_1^3 + \Big[6y + \frac{5y^2}{2} - \frac{y^3}{3}\Big]_{-1}^1 = \frac{34}{3} + \frac{34}{3} = \frac{68}{3}.$$

POLAR COORDINATES

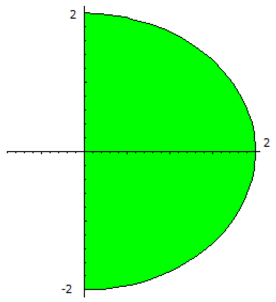
For any point P other than the origin, let r be the distance between P and the origin, and φ an angle having its initial side on the positive x axis and its terminal side on the line segment joining P and the origin. The pair (r, φ) is called a set of polar coordinates for the point P. Every point (x, y) in the plane has both Cartesian and polar coordinates (r, φ):

$$\begin{cases} x &= r\cos\varphi \\ y &= r\sin\varphi \end{cases}.$$



We have the following result for polar coordinates:

$$\int_D\int f(x,y)\,dxdy = \int_\Delta\int f(r\cos\varphi, r\sin\varphi)\,r\,drd\varphi\,.$$



Example 6. Using polar coordinates, calculate $\iint\limits_D xy^2\,dxdy$ where $D : x^2 + y^2 \le 4,\ x \ge 0$.

Solution: The region of integration is a semicircle with radius equal 2. Therefore, the region in polar coordinates is given by $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$.

After substituting x and y with polar coordinates, we have:

$$\begin{aligned}\iint\limits_D xy^2\,dxdy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\int_0^2 (r\cos\theta)\cdot(r\sin\theta)^2 r\,dr\Big)\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\int_0^2 r^4\sin^2\theta\cos\theta dr\Big)\,d\theta \\ &= \Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\cos\theta d\theta\Big)\cdot\Big(\int_0^2 r^4 dr\Big) = [\frac{\sin^3\theta}{3}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot[\frac{r^5}{5}]_0^2 = \frac{64}{15}.\end{aligned}$$

Example 7. Using polar coordinates, calculate $\iint\limits_D (x^2+y^2)\,dxdy$, where $D : x^2 + y^2 - 2y \le 0$.

Solution (a): Let us represent the equation describing D in a different form:

$$x^2 + y^2 - 2y \le 0$$
$$x^2 + (y^2 - 2y + 1) - 1 \le 0$$
$$x^2 + (y-1)^2 \le 1$$

Such an equation describes a circle with the origin in $(0,1)$, so we cannot describe it with polar coordinates as easily as in **Example 6**. Let us substitute $x = r\cos\theta$ and $y = r\sin\theta$:

$$x^2 + y^2 - 2y \leq 0$$
$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta \leq 0$$
$$r \leq 2\sin\theta$$

the integral is equal to:

$$\iint\limits_D (x^2+y^2)\,dxdy = \int_0^\pi\Big(\int_0^{2\sin\theta} r^2(\sin^2\theta+\cos^2\theta)rdr\Big)\,d\theta = \int_0^\pi(\int_0^{2\sin\theta} r^3dr)d\theta = \int_0^\pi [\frac{r^4}{4}]_0^{2\sin\theta}d\theta$$
$$= 4\int_0^\pi \sin^4\theta d\theta = 4[\frac{3\theta}{8} - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}]_0^\pi = \frac{3\pi}{2}.$$

Angle θ ranges from 0 to only π, because for $\theta \in (\pi, 2\pi]$ the radius would be negative – which is impossible.

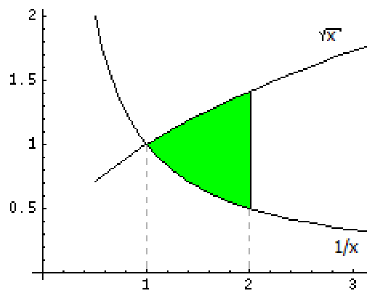
Solution (b): Since the circle is moved by a vector of $\vec{v} = (0,1)$, then we can also move the function x^2+y^2 by the same vector. The new function will be $x^2+(y-1)^2$. We can now use the method from **Example 6**:

$$\iint\limits_D (x^2+y^2)\,dxdy = \int_0^{2\pi}\Big(\int_0^1 (r^2\cos^2\theta + (r\sin\theta - 1)^2)rdr\Big)d\theta = \cdots = \frac{3\pi}{2}.$$

Area of a bounded region in the plane

The area of a closed bounded plane region R is given by the formula

$$Area = \iint\limits_R 1\,dxdy.$$



Example 8. Calculate the area of a region bounded by curves $y = \frac{1}{x}$, $y = \sqrt{x}$ and a line $x = 2$. Sketch the region.

Solution: The area is equal to:

$$\int_1^2(\int_{y=\frac{1}{x}}^{y=\sqrt{x}} 1\,dy)\,dx = \int_1^2 [y]_{y=\frac{1}{x}}^{y=\sqrt{x}}\,dx = \int_1^2(\sqrt{x} - \frac{1}{x})\,dx = [\frac{2}{3}x^{\frac{3}{2}} - \ln|x|]_1^2 = \frac{1}{3}(-2+4\sqrt{2}-\ln 8).$$

Volume

Let R be a a bounded region in the OXY plane and f be a function continuous on R. If f is nonnegative and integrable on R, then the volume of the solid region between the graph of f and R is given by

$$Volume = \iint\limits_R f(x, y)\, dxdy.$$

Let R be a a bounded region in the xy plane and g_1, g_2 be continuous functions on R. If g_1 and g_2 are integrable on R such that $g_1(x, y) \leq g_2(x, y)$, then the volume of the solid region between the graph of g_1 and g_2 is given by

$$Volume = \iint\limits_R (g_2(x, y) - g_1(x, y))\, dxdy.$$

Example 9. Calculate the volume of a solid bounded by curves $y = x^2,\ y = 1,\ z = 0,\ z = 2y$.

solution: The region of integration is bounded by $y = x^2$ and $y = 1$ and $f(x, y) = 2y$. Therefore:

$$\begin{aligned} Volume &= \int\limits_{x=-1}^{x=1} \Big(\int\limits_{y=x^2}^{y=1} 2y\, dy\Big)\, dx = \int\limits_{x=-1}^{x=1} [y^2]_{y=x^2}^{y=1}\, dx = \int\limits_{x=-1}^{x=1} (1 - x^4)\, dx = [x - \frac{x^5}{5}]_{-1}^{1} \\ &= 1 - \frac{1}{5} - (-1 + \frac{1}{5}) = 2 - \frac{2}{5} = \frac{8}{5}. \end{aligned}$$

Surface

Let S be the surface $z = f(x, y)$ where the points (x, y) come from the given region R in the OXY plane. Then

$$Area_S = \iint\limits_R \sqrt{1 + (\tfrac{\partial f}{\partial x})^2 + (\tfrac{\partial f}{\partial y})^2}\, dxdy,$$

where f and its first partial derivatives are continuous.

Example 10. Calculate the surface of a plane $2x + 2y + z = 8$ bounded by the coordinate system axes.

Solution: After transformations of the equation of a plane, we have $\frac{x}{4} + \frac{y}{4} + \frac{z}{8} = 1$, so the plane intersects the coordinate system axes at points $A = (4, 0, 0)$, $B = (0, 4, 0)$ and $C = (0, 0, 8)$. Therefore, the region of integration is bounded by $x = 0,\ y = 0,\ y = -x + 4$. We also have

$f(x,y) = z = 8 - 2x - 2y$, so $\frac{\partial f}{\partial x} = -2$ and $\frac{\partial f}{\partial y} = -2$. Therefore:

$$Surface = \int_{x=0}^{x=4} (\int_{y=0}^{y=-x+4} \sqrt{1+(-2)^2+(-2)^2}\ dy)\ dx = \int_{x=0}^{x=4} (\int_{y=0}^{y=-x+4} \sqrt{9}\ dy)\ dx = 3 \int_{x=0}^{x=4} (\int_{y=0}^{y=-x+4} 1\ dy)\ dx$$

$$= 3 \int_{x=0}^{x=4} [y]_{y=0}^{y=-x+4}\ dx = 3 \int_{x=0}^{x=4} (-x+4)\ dx = 3[-\frac{x^2}{2} + 4x]_0^4 = 24.$$