Proper limit at a point *(Heine's definition)*

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function. Function f has a proper limit g at a point (x_0, y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = g$$

if and only if for every sequence (x_n, y_n) such that $\lim_{n\to\infty}(x_n, y_n) = (x_0, y_0)$ we have

$$\lim_{n\to\infty} f(x_n, y_n) = g.$$

Improper limit at a point *(Heine's definition)*

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function. Function f has an improper limit $\pm\infty$ at a point (x_0, y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \pm\infty$$

if and only if for every sequence (x_n, y_n) such that $\lim_{n\to\infty}(x_n, y_n) = (x_0, y_0)$ we have

$$\lim_{n\to\infty} f(x_n, y_n) = \pm\infty.$$

Remark

Heine's definition of a limit may be easily used to show that a limit does not exist – it is enough to find two different sequences (x'_n, y'_n) and (x''_n, y''_n) and show that $f(x'_n, y'_n)$ tends to a different number than $f(x''_n, y''_n)$.

Limits That Arise Frequently

Once you substitute sequences for x and y, you will need to use some commonly used formulas:

- $$\lim_{n\to\infty} q^n = \begin{cases} \text{does not exist} & q \leq -1 \\ 0 & |q| < 1 \\ 1 & q = 1 \\ \infty & q > 1 \end{cases}.$$
- If $a > 0$, then $\lim_{n\to\infty} \sqrt[n]{a} = 1$.
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

- If $\lim\limits_{n\to\infty} a_n = \infty$, then $\lim\limits_{n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$.

- $\lim\limits_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$.

Examples

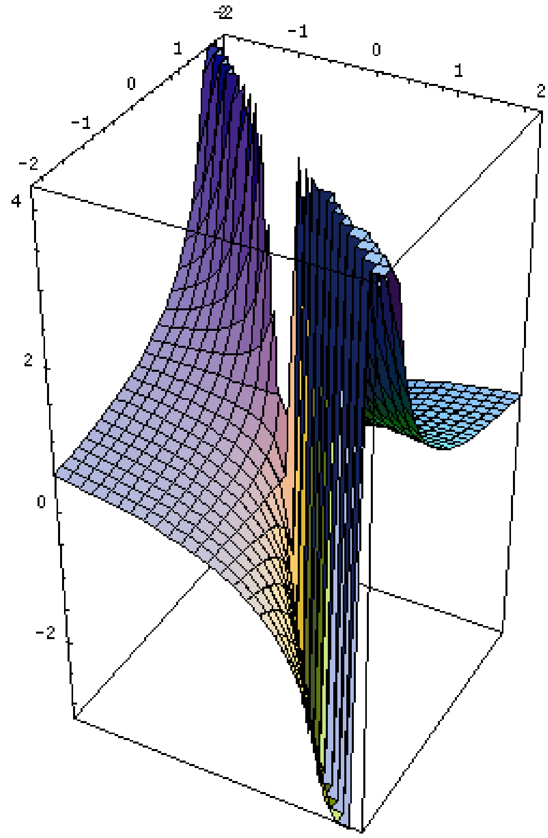
Example 1. Show that a limit $\lim\limits_{(x,y)\to(0,0)} \frac{x}{x+y}$ does not exist.

Solution: We will find two sequences that both tend to $(0,0)$, but for which function $f(x,y) = \frac{x}{x+y}$ approaches different numbers. Let the sequences be $(x'_n, y'_n) = (\frac{1}{n}, 0)$ and $(x''_n, y''_n) = (0, \frac{1}{n})$.

Now, let us calculate two limits:

- $\lim\limits_{n\to\infty} f(x'_n, y'_n) = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}+0} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1$.
- $\lim\limits_{n\to\infty} f(x''_n, y''_n) = \lim\limits_{n\to\infty} \frac{0}{0+\frac{1}{n}} = 0$.

Obviously $1 \neq 0$, so it is not true that for every sequence (x_n, y_n) function $f(x_n, y_n)$ tends to the same number as n approaches infinity. Therefore, function $f(x,y) = \frac{x}{x+y}$ does not have a limit at a point $(0,0)$.



We can informally check our result by plotting this function in the neighbourhood of $(0,0)$, say in $[-2,2]\times[2,2]$. We can see that the plot of this functions has two "wings" that spring in two different directions in the neighbourhood of $(0,0)$.

Example 2. Show that a limit $\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ does not exist.

Solution: We will find two sequences that both tend to $(0,0)$, but for which function $f(x,y) = \frac{x}{x+y}$ approaches different numbers. Let the sequences be $(x'_n, y'_n) = (\frac{1}{n}, \frac{3}{n})$ and $(x''_n, y''_n) = (\frac{1}{n}, \frac{2}{n})$.
Now, let us calculate two limits:

- $\lim\limits_{n\to\infty} f(x'_n, y'_n) = \lim\limits_{n\to\infty} \frac{2\cdot\frac{1}{n}\cdot\frac{3}{n}}{(\frac{1}{n})^2+(\frac{3}{n})^2} = \lim\limits_{n\to\infty} \frac{\frac{6}{n^2}}{\frac{1}{n^2}+\frac{9}{n^2}} = \frac{6}{1+9} = \frac{6}{10} = \frac{3}{5}$.

- $\lim\limits_{n\to\infty} f(x''_n, y''_n) = \lim\limits_{n\to\infty} \frac{2\cdot\frac{1}{n}\cdot\frac{2}{n}}{(\frac{1}{n})^2+(\frac{2}{n})^2} = \lim\limits_{n\to\infty} \frac{\frac{4}{n^2}}{\frac{1}{n^2}+\frac{4}{n^2}} = \frac{4}{1+4} = \frac{4}{5}$.

Again, we obtained two different limits, which implies that a limit does not exist.

Example 3. Show that a limit $\lim\limits_{(x,y)\to(0,1)} \frac{x^6}{y^3-1}$ does not exist.

Solution: Let $(x'_n, y'_n) = (\frac{1}{n}, \sqrt[3]{1-\frac{1}{n^3}})$ and $(x''_n, y''_n) = (\frac{1}{\sqrt{n}}, \sqrt[3]{1-\frac{1}{n^3}})$.

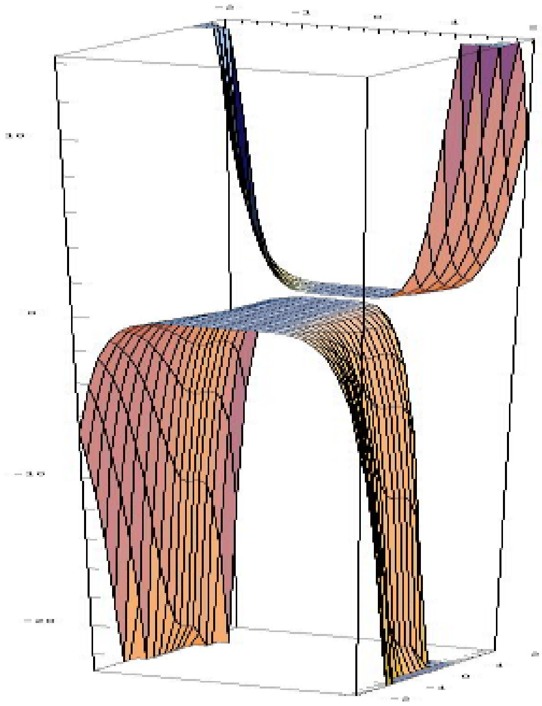
It is obvious that $\lim\limits_{n\to\infty} \sqrt[3]{1-\frac{1}{n^3}} = 1$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0 = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}$.

We have the following result:

- $\lim\limits_{n\to\infty} f(x'_n, y'_n) = \lim\limits_{n\to\infty} \frac{\frac{1}{n^6}}{(1-\frac{1}{n^3})-1} = \lim\limits_{n\to\infty} \frac{\frac{1}{n^6}}{-\frac{1}{n^3}} = = \lim\limits_{n\to\infty} -\frac{n^3}{n^6} = 0.$
- $\lim\limits_{n\to\infty} f(x''_n, y''_n) = \lim\limits_{n\to\infty} \frac{\frac{1}{n^3}}{(1-\frac{1}{n^3})-1} = \lim\limits_{n\to\infty} \frac{\frac{1}{n^3}}{-\frac{1}{n^3}} = = -1.$

Since $0 \neq -1$, the limit does not exist.

The figure on the right shows the plot of $f(x,y) = \frac{x^6}{y^3-1}$ for $(x,y) \in [-2,2] \times [-2,2]$.



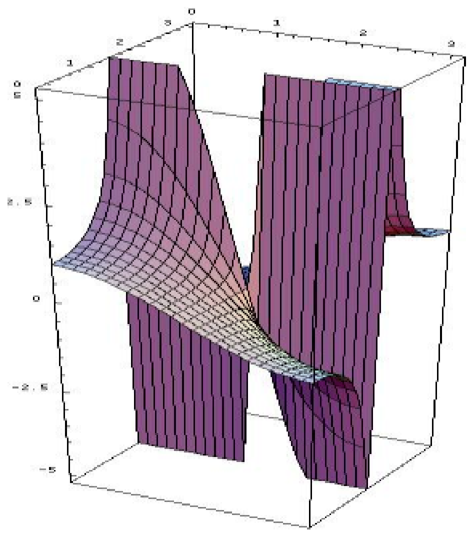
Example 4. Show that a limit $\lim\limits_{(x,y)\to(0,0)} \frac{x}{y}$ does not exist.

Solution: Let $(x'_n, y'_n) = (\frac{1}{n}, \frac{1}{n})$ and $(x''_n, y''_n) = (\frac{1}{n}, \frac{2}{n})$. Then, $\lim\limits_{n\to\infty} \frac{x'_n}{y'_n} = 1$ and $\lim\limits_{n\to\infty} \frac{x''_n}{y''_n} = \frac{1}{2}$. Thus, the limit does not exist.

Example 5. Show that a limit $\lim\limits_{(x,y)\to(\frac{\pi}{2},\frac{\pi}{2})} \frac{\cos x}{\cos y}$ does not exist.

Solution: Let us "move" the cosines by $\frac{\pi}{2}$ to obtain the sines (which we are more used to). Since $\cos(x+\frac{\pi}{2}) = -\sin x$, we now need to show that $\lim\limits_{(x,y)\to(0,0)} \frac{\sin x}{\sin y}$ does not exist. We will also need to use the fact, that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. So:

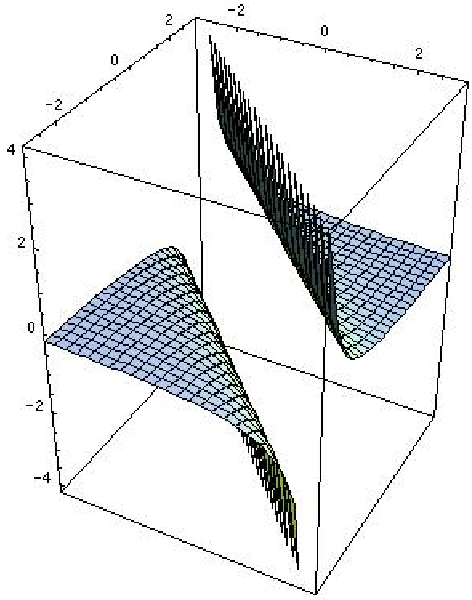
$\lim\limits_{(x,y)\to(0,0)} \frac{\sin x}{\sin y} = \lim\limits_{(x,y)\to(0,0)} \frac{x \cdot \frac{\sin x}{x}}{y \cdot \frac{\sin y}{y}} = \lim\limits_{(x,y)\to(0,0)} \frac{x \cdot 1}{y \cdot 1} = \lim\limits_{(x,y)\to(0,0)} \frac{x}{y},$



and it has already been shown in **Example 4** that such a limit does not exist.

The figure shows the plot of $f(x,y) = \frac{\cos x}{\cos y}$ for $(x,y) \in [0,\pi] \times [0,\pi]$.

Example 6. Show that a limit $\lim_{(x,y)\to(0,0)} \frac{1}{x+y}$ does not exist.



Solution: We need to notice that:

- $\lim_{(x,y)\to(0^+,0^+)} \frac{1}{x+y} = [\frac{1}{+0}] = \infty$,
- $\lim_{(x,y)\to(0^-,0^-)} \frac{1}{x+y} = [\frac{1}{-0}] = -\infty$.

Therefore, the limit does not exist.

Figure on the right shows the plot of $f(x,y) = \frac{1}{x+y}$.

Example 7. By considering different lines of approach show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution: The domain of function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ is $\mathbf{R}^2 \setminus\{(0,0)\}$. We will approach point $(0,0)$ along two lines: $y = x$ and $y = 2x$:

a) along $y = x$:

$$f(x,y) = f(x,x) = \frac{x^2-x^2}{x^2+x^2} = \frac{0}{2x^2},$$
$$\lim_{(x,y)\to(0,0)\ along\ y=x} f(x,y) = 0,$$

b) along $y = 2x$:

$$f(x,y) = f(x,2x) = \frac{x^2-4x^2}{x^2+4x^2} = \frac{-3x^2}{5x^2} = -\frac{3}{5},$$
$$\lim_{(x,y)\to(0,0)\ along\ y=2x} f(x,y) = -\frac{3}{5}.$$

Limits presented in poiunts (a) and (b) are different, therefore $f(x,y)$ has no limit as (x,y) tends to $(0,0)$.