Definition

A <u>function of several variables</u> consists of two parts: a <u>domain</u>, which is a collection of points in the plain or in space, and a <u>rule</u>, which assigns to each member of the domain one and only one number.

A function of several variables is called a <u>function of two variables</u> if its domain is a set of points in the plane and a <u>function of three variables</u> if its domain is a set of points in space.

The graph of a function f of two variables is the collection of points $P(x, y, f(x, y))$ for which (x, y) is in the domain of f.

Combinations of Functions of Several Variables

- $(f \pm g)(x, y) = f(x, y) \pm g(x, y)$
- $(f \cdot g)(x, y) = f(x, y) \cdot g(x, y)$
- $\left(\dfrac{f}{g}\right)(x, y) = \dfrac{f(x, y)}{g(x, y)}$

Example

Sketch the graph of z:

a) $x^2 + y^2 = 1$

b) $x^2 + y^2 + z^2 = 1$

c) $z = x^2 + y^2$

d) $z^2 = x^2 + y^2$

e) $z = x^2$

Example

Sketch the solid region bounded by:

a) $x^2 + y^2 = 1$, $z = -1$, $z = 4$

b) $z = \sqrt{1 - x^2 - y^2}$, $z = 0$

c) $z = \sqrt{x^2 + y^2}$, $z = 4$

d) $z = x^2 + y^2$, $z = 4$

e) $z^2 = x^2 + y^2$, $z = -1$, $z = 1$

f) $x^2 + y^2 = 1$, $z = -1$, $z = y + 1$

g) $z = x^2$, $z = 1$, $y = 4$, $y = -4$

h) $z = 6 - x^2 - y^2$, $z = \sqrt{x^2 + y^2}$

i) $x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$

j) $x^2 - 2x + y^2 = 0$, $z = 0$, $z = 10$

Definition

Let f be definied throughout a set containing a disc centred at (x_0, y_0) except possibly at (x_0, y_0) itself, and let g be a number. The limit of $f(x, y)$ as $(x, y) \to (x_0, y_0)$ is the number g if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all points $f(x, y)$ as $(x, y) \neq (x_0, y_0)$ in the domain of f:

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |f(x, y) - g| < \varepsilon\ .$$

If the values of a function $z = f(x, y)$ can be made as close as we like to a fixed number g by taking the point (x, y) close to the point (x_0, y_0), but not equal to (x_0, y_0), we say that g is the limit of f as (x, y) approches (x_0, y_0). In symbols, we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = g$$

and we say "the limit of f as (x, y) approches to (x_0, y_0) equals g".

The Limit Combination Theorem

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = g_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x, y) = g_2$, then

- $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x, y) \pm g(x, y)) = g_1 \pm g_2$,

- $\lim\limits_{(x,y)\to(x_0,y_0)} cf(x, y) = cg_1$,

- $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x, y)g(x, y)) = g_1 \cdot g_2$,

- $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{g_1}{g_2}$ if $g_2 \neq 0$.

The formulas for limits of functions of three variables are similar.

Example

Find the limits:

a) $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2+y^2}{x+y+1}$;

b) $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2-y^2}{x-y}$, $x \neq y$;

c) $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x-y}$;

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x^2+y^2}$.

Example

By considering different lines of approach, show that the functions have no limit as $(x,y)\to(0,0)$:

a) $f(x,y) = \dfrac{x+y}{x-y}$, b) $f(x,y) = \dfrac{x^2y}{x^4+y^2}$.

Definition

A function $f(x,y)$ is said to be continuous at the point (x_0,y_0) if

- f is defined at (x_0,y_0),
- $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and
- $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

Theorem

If f and g are both continuous at a point, then their sum $f+g$ is contiuous at that point. Similar results hold for differences, products, and multiples of continuous functions. Also, the quotient of two continuous functions is continuous wherever it is defined.

Example

Show that the function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ is continuous at every point except the origin.

Example

Show that f is not continuous at $(0,0)$:

a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$,

b) $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^{12}+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Continuity on a Set

Let R be a set in the plane. Then for any point P in R, one of the following condition holds:

1. There is a disc centered at P and contained in R.
2. Every disc centered at P contains points outside R.

Points satisfying condition 1. are <u>interior points</u> of R, and points satisfying condition 2. are <u>boundary points</u> of R.

We say that f is <u>continuous at a boundary point (x_0, y_0)</u> of R if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

and f is <u>continuous on a set R</u> containing its boundary if it is continuous at each point of R.

Partial Derivatives

Let f be a function of two variables. If we fix one of the two variables, say $y = y_0$, the function whose values are $f(x, y_0)$ is a function of x alone. If that function has a derivative at x_0, we call the derivative a <u>partial derivative</u> at (x_0, y_0).

Partial derivatives of f are frequently denoted

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

and

$$f_x, \quad f_y.$$

Definition

Let f be a function of two variables and let (x_0, y_0) be in domain of f.

The <u>partial derivative of f with respect to x at (x_0, y_0)</u> is defined by

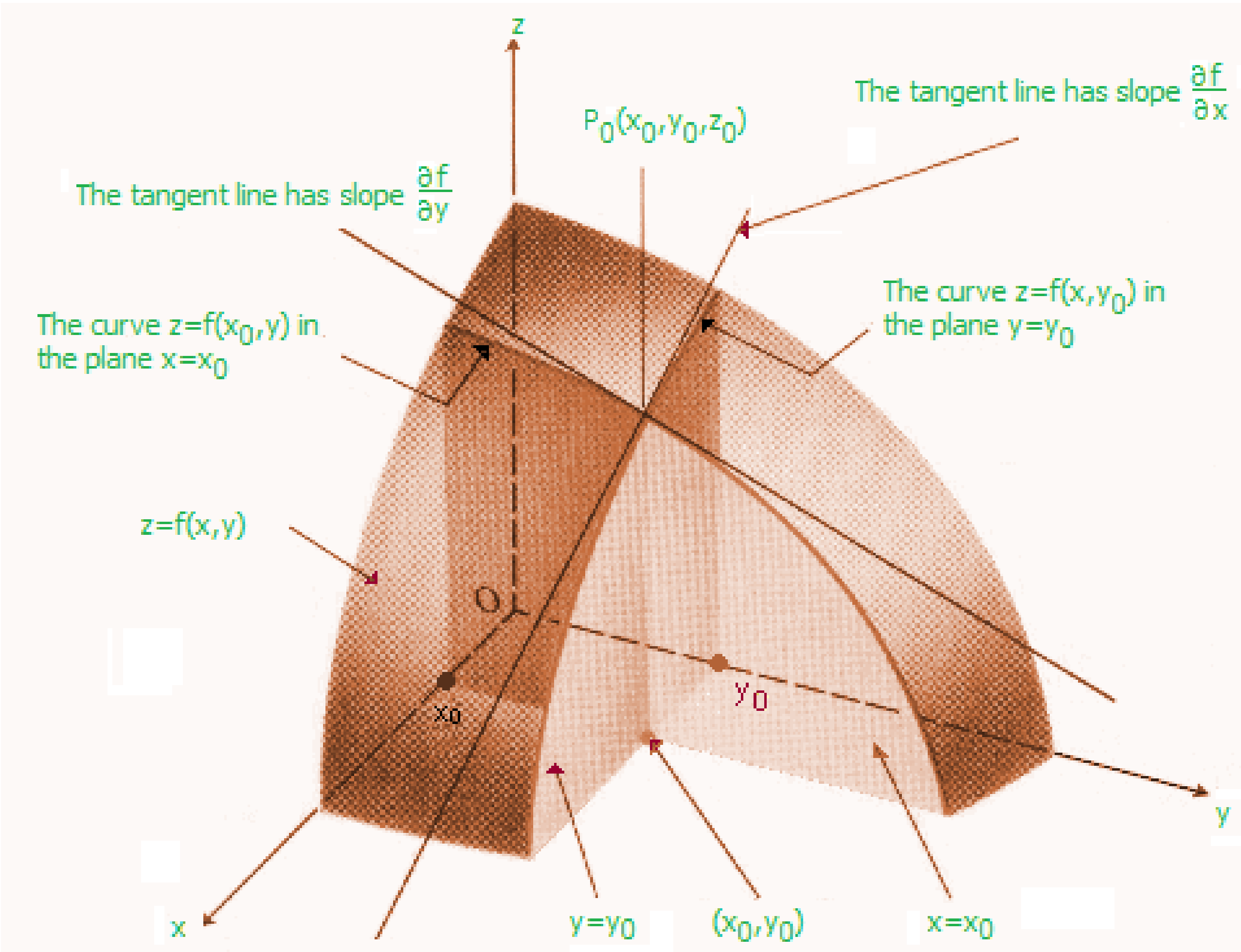
$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

provided that this limit exists.

The <u>partial derivative of f with respect to y at (x_0, y_0)</u> is defined by

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{\Delta y\to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

provided that this limit exists.



The plane $y = y_0$ cuts the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$. At each x, the slope of this curve is $\dfrac{\partial f}{\partial x}(x, y_0)$.

The definition of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. A partial derivative of a function of several variables is its derivative with respect to one of those variables with the others held constant.

Second partial derivatives are defined to be partial derivatives of first partial derivatives, and higher derivatives are similarly defined. If both of the first order partial derivatives exist in a neighborhood (x_0, y_0) and they are functions of x and y, then we can differentiate each with respect to x or y:

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} = \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \ , \ \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right).$$

Partial derivatives involving more than one variable are called mixed partial derivatives.

NOTATION

Pure second partial derivatives: $f_{xx} \equiv \dfrac{\partial^2 f}{\partial x^2}$, $f_{yy} \equiv \dfrac{\partial^2 f}{\partial y^2}$.

Mixed partial derivatives: $f_{xy} \equiv \dfrac{\partial^2 f}{\partial x \partial y}$, $f_{yx} \equiv \dfrac{\partial^2 f}{\partial y \partial x}$.

THE MIXED DERIVATIVE THEOREM

If $f(x, y)$ and its partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ are definied in a region containing a point (a, b) and are all continuous at (a, b), then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b) \ .$$

THE CHAIN RULE

INTRODUCTION

When we are interested in the temperature $f(x, y, z)$ at points on the path

$$x = x(t) \ , \quad y = y(t) \ , \quad z = z(t)$$

in space or in the preassure or density along the path through a gas of fluid, we may think of f as a function of the single variable t. At each t the temperature at the point $(x(t), y(t), z(t))$ on the path is the value of the composite $f(x(t), y(t), z(t))$. If we then wish to know the rate at which f changes with respect to t along the path, we have only to differentiate this composite with respect to t (provided that $\dfrac{df}{dt}$ exists).

THE CHAIN RULE FOR FUNCTION OF TWO AND THREE VARIABLES

If $z = f(x, y)$ and its partial derivatives are continuous and $x = x(t)$, $y = y(t)$ are differentiable functions of t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \ .$$

If $w = f(x, y, z)$ and its partial derivatives are continuous and $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable functions of t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \ .$$

Definition of Gradient

- If the partial derivative of $f(x, y, z)$ are definied at $P(x_0, y_0, z_0)$, then the gradient of f at P is the vector
$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$
obtained by evaluating the partial derivatives of f at P.

- The two-variable formulas are obtained by dropping the z-terms from the three-variable formulas:
$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \; .$$

Another notation for the gradient of f is *gradf* read the way it is written. The symbol ∇f is read "grad f" as well as "gradient f" and "del f".

Tangent Planes

The graph of a function $f(x, y)$ is a surface in $\mathbf{R}^3$ (three dimensional space) and so we can now start thinking of the plane that is "tangent" to the surface at the point.
If $f(x, y)$ and its partial derivatives are all continuous at (x_0, y_0), then we define the tangent plane of the surface at the point $(x_0, y_0, f(x_0, y_0))$:

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \; .$$

Differentials

If f is function of two variables, which is differentiable at (x, y) (in the domain of f), then

$$f(x + h, y + h) = f(x, y) + \frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x, y)k \; .$$

The number
$$\frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x, y)k$$
is called the differential (or total differential) of f at (x, y) (with increments h and k) and is denoted
$$df \; .$$

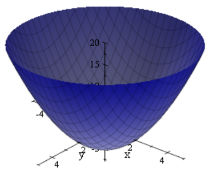
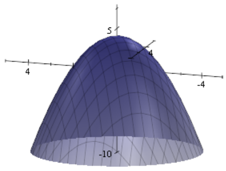
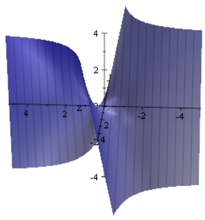
Thus df depends on x, y, h and k. We can write this formula as

$$dy = \frac{\partial f}{\partial x}(x, y)dx + \frac{\partial f}{\partial y}(x, y)dy \; .$$

Examples of Partial Differential Equations from Physics

- The One-dimensional Heat Equation (= Diffusion Equation = Telegraph Equation)
- The Three-dimensional (The Two-dimensional) Laplace Equation
- The Wave Equation

Minima, Maxima and Saddle Points

For a function $f(x, y)$ of two independent variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent plane. At such points we then look for a local maximum, a local minimum, or a local saddle point.

We organize the search for the local extreme values assumed by a continuous function $f(x, y)$ into two steps:

- The local maxima and minima of f can occur only at points where

$$\frac{\partial f}{\partial x} = 0 \quad \wedge \quad \frac{\partial f}{\partial y} = 0$$

  and points where $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ fail to exist. We call these the critical points of f.

- If f and its first and second partial derivatives are continuous, there is the second derivative test that may identify the behavior of f at critical point (a, b). The expression

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

  is called the discriminant of f and the test goes like this:
  If $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$, then
  i) if $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$, then f has a local maximum at (a, b)
  ii) if $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$, then f has a local minimum at (a, b)

iii) if $D(a, b) < 0$, then f has a local saddle point at (a, b)

iv) if $D(a, b) = 0$, then the test is inconclusive at (a, b) - we must find some other way to determine the behavior of f at (a, b).

Maximum-Minimum Theorem for two variables

Let R be a bounded set in the plane that contains its boundary, and let f be a continuous on R. Then f has both a maximum and minimum value on R.

We organize the search for the absolute maximum and minimum values into three steps:

- 1. Find the critical points of f in R, and compute the values of f at this points.
- 2. Find the extreme values of f on the boundary of R.
- 3. The maximum value of f on R will be a largest of the values computed in steps 1. and 2., and the minimum value of f on R will be the smallest of those values.