1 Exponential function and its properties

The power function was defined earlier as $f(x) = x^a$, where a was a given real number, and x was the variable. If we interchange these two letters, we will have a power a^x with a fixed base and variable exponent. Such a function is called an exponential function.

> **Def. 1.1** The **exponential function f with base a** is defined by
>
> $$f(x) = a^x$$
>
> where a is a real number such that $a > 0$, $a \neq 1$ and x is any real number.

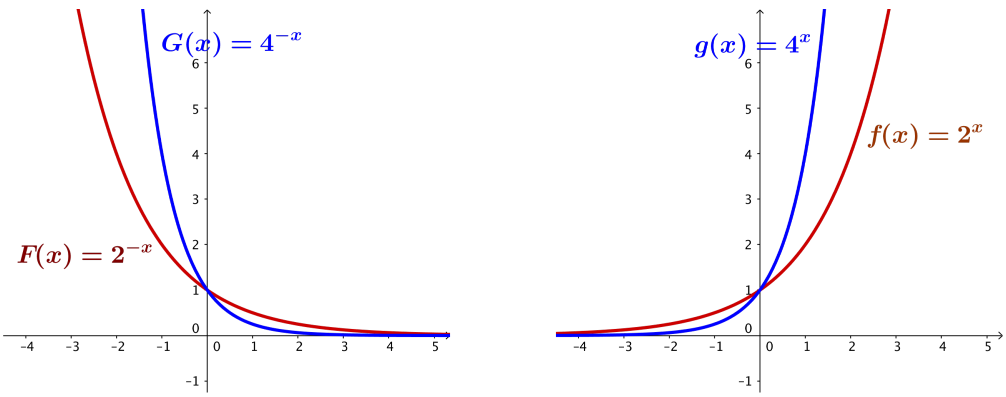
Your first question, as always, should be: why the assumptions on base a ? Here is why:

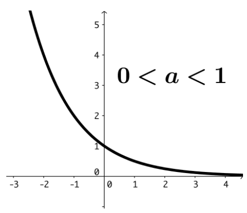
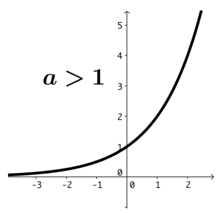
- $a = 1$ is excluded because it yields $f(x) = 1^x = 1$, which is a constant linear function.
- Similarly, $a = 0$ leads to $f(x) = 0^x$, which equals 0 for all $x > 0$.
- $a > 0$, because for example when $a = -1$, a^x doesn't exist for every x

$$f\left(\frac{1}{3}\right) = \sqrt[3]{-1} = -1, \quad f\left(\frac{1}{2}\right) = \sqrt{-1} \notin I\!R$$

If we have a function, then we can draw its graph. And **we can read most of the properties of a function from its graph.** Exponential functions come in two varieties,

- if the base a is larger than 1, e.g. 4^x then the values will grow with increasing exponent x,
- if the base a is smaller than 1, e.g. $\left(\frac{1}{4}\right)^x = 4^{-x}$ then the values will decline as the exponent x increases.



Properties of the exponential function $f(x) = a^x$	
 $0 < a < 1$ decreasing	 $a > 1$ increasing
one-to-one $f(0) = 1$ domain: $I\!R$ range: $I\!R_+$	

• Exponential equations

$$\boxed{\textbf{one-to-one property: } a^x = a^y \quad \Leftrightarrow \quad x = y}$$

The one-to-one property is used when solving exponential equations.

Example 1.1 To explain the idea of using one-to-one property, let's solve this simple equation

$$\left(\frac{1}{27}\right)^{4x} = 81$$

As always, start with determining restrictions on the variable x. Here there are none, so $x \in I\!R$. To solve the equation, first we try to express both sides of the equation as powers with the same base

$$\left(3^{-3}\right)^{4x} = 3^4$$

$$3^{-12x} = 3^4$$

and then, by the one-to-one property, we may drop the common base

$$-12x = 4 \quad \Longleftrightarrow \quad \boxed{x = -\frac{1}{3}}$$

—— ✓ ✓ ✓ ——

Example 1.2 The same approach is the first thing we try with more complicated equations as well.

$$7 \cdot 4^x - 26 = 2^{2x+1} + 7 \cdot 4^{x-1}$$

There are no restrictions on x, so $x \in I\!R$. We simplify the equation,

$$7 \cdot 4^x - 26 = 2^{2x} \cdot 2^1 + 7 \cdot 4^x \cdot 4^{-1}$$

$$7 \cdot 4^x - 2 \cdot 4^x - \frac{7}{4} \cdot 4^x = 26$$

$$\frac{13}{4} \cdot 4^x = 26$$

$$4^x = 8$$

$$2^{2x} = 2^3 \quad \Longleftrightarrow \quad 2x = 3 \quad \Longleftrightarrow \quad \boxed{x = \frac{3}{2}}$$

—— ✓ ✓ ✓ ——

Example 1.3 It is, of course, not always possible to express the entire expression as one power. In such cases we may try to use substitution.

$$3^{2x} - 8 \cdot 3^x - 9 = 0$$

As before, $x \in I\!R$. By using the substitution $3^x = t > 0$, we change the exponential equation into a quadratic one

$$t^2 - 8t - 9 = 0$$

$$(t+1)(t-9) = 0$$

This equation has two solutions, but only the positive one is valid

$$t = -1 < 0, \quad \vee \quad t = 9$$

Now, we reverse the substitution

$$3^x = 9$$

$$3^x = 3^2 \quad \Longrightarrow \quad \boxed{x = 2}$$

—— ✓ ✓ ✓ ——

Example 1.4 Here is one example with some initial restrictions on x.

$$\frac{1}{8} \cdot 4^{\sqrt{x}} = 8^{x-1}$$

We start with noting that there is a restriction on x due to the square root, that is: $x \geq 0$. Then, we simplify

$$2^{-3} \cdot 2^{2\sqrt{x}} = 2^{3(x-1)}$$

$$2^{-3+2\sqrt{x}} = 2^{3x-3}$$

and by the one-to-one property, we may drop the common base

$$-3 + 2\sqrt{x} = 3x - 3$$
$$2\sqrt{x} = 3x$$

We may square both sides of this equation since they are both nonnegative ($x \geq 0$)

$$4x = 9x^2$$

$$x(9x-4) = 0 \quad \Longrightarrow \quad \boxed{x = 0 \ \vee \ x = \frac{4}{9}}$$

—— ✓ ✓ ✓ ——

- Exponential inequalities

When solving inequalities with an exponential function we most often use the monotonicity property.

Monotonicity

$$a^x \leq a^y \iff x \leq y, \quad \text{for } a > 1$$
$$a^x \leq a^y \iff x \geq y, \quad \text{for } 0 < a < 1$$

To illustrate the idea we'll start with a similar example we used to explain how to solve equations.

Example 1.5

$$\left(\frac{1}{27}\right)^{4x} \geq 81$$

From this, just as with the equation, we get

$$3^{-12x} \geq 3^4$$

Since the exponential function with base 3 is an increasing function, we drop the base without changing the direction of the inequality

$$-12x \geq 4 \quad \Longrightarrow \quad \boxed{x \leq -\frac{1}{3}}$$

Notice that we could have simplified the original inequality to

$$\left(\frac{1}{3}\right)^{12x} \geq \left(\frac{1}{3}\right)^{-4}$$

And here, since the exponential function with base $\frac{1}{3}$ is a decreasing function, when dropping the base we'd have to change the direction of the inequality

$$12x \leq -4$$

which, of course, leads to the same solution.

—— ✓ ✓ ✓ ——

Example 1.6 Substitution is useful in cases where we're unable to simplify the entire expression into one power

$$16^x - 8 < -2^{2x+1}$$

There are no restrictions on x, so $x \in \mathbb{R}$.

$$4^{2x} + 2 \cdot 4^x - 8 < 0$$

We make the substitution $4^x = t > 0$, and change the exponential inequality into a quadratic one,

$$t^2 + 2t - 8 < 0$$

$$(t-2)(t+4) < 0$$

$$-4 < t < 2 \quad \wedge \quad t > 0 \quad \Longrightarrow \quad 0 < t \leq 2$$

Now, we reverse the substitution

$$0 < 4^x \leq 2$$

$$2^{2x} \leq 2 \quad \Longrightarrow \quad 2x < 1 \quad \Longrightarrow \quad \boxed{x \leq \frac{1}{2}}$$

—— ✓ ✓ ✓ ——

Example 1.7 Here is an example that at first looks complicated

$$7 + 4^{\frac{2\sqrt{x-1}-1}{2}} > 4^{\sqrt{x-1}+1}$$

Start with restrictions on x.

$$x - 1 \geq 0 \quad \Longrightarrow \quad x \geq 1$$

Then simplify the inequality,

$$7 > 4 \cdot 4^{\sqrt{x-1}} - 4^{-\frac{1}{2}} \cdot 4^{\sqrt{x-1}}$$

$$7 > \frac{7}{2} \cdot 4^{\sqrt{x-1}}$$

$$2 > 2^{2\sqrt{x-1}}$$

$$1 > 2\sqrt{x-1}$$

Since both sides of the inequality are positive, we may square it

$$1 > 4(x-1) \quad \Longrightarrow \quad x < \frac{5}{4} \quad \wedge \quad x \geq 1 \quad \Longrightarrow \boxed{1 \leq x < \frac{5}{4}}$$

—— ✓ ✓ ✓ ——

References

[1] *Matematyka – podstawy z elementami matematyki wyszej*, edited by B. Wikieł , PG publishing house, 2009.