1 Functions. Definition and graphs

Def. 1.1 Given two nonempty sets $X, Y \subset \mathbb{R}$, a **function** f is a rule that assigns to each value $x \in X$ a *unique* value $y = f(x) \in Y$. We write,

$$f : X \longrightarrow Y, \quad y = f(x) \;\text{ for }\; x \in X$$

- $X = D_f$ is called the **domain** of the function
- **Natural domain** is the set of those $x \in \mathbb{R}$, for which the formula of the function makes sense.
- The set of possible values of the function is called the **range**, $R_f = \{y \in Y : y = f(x), x \in X\} \subset Y$
- The **argument** of the function is the expression on which the function works. For example, z is the argument in $f(z)$, 5 is the argument in $f(5)$, $x^2 - 6$ is the argument in $f(x^2 - 6)$.
- The **independent variable** is the variable associated with the domain (x in the definition above), and the **dependent variable** belongs to the range (y in the definition above).

Example 1.1 Find the domain of each function.

- $f(x) = \sqrt{9 - x^2}$
  We cannot have a negative argument under the square root, so
  $$9 - x^2 \geq 0 \quad \Leftrightarrow \quad (3 - x)(3 + x) \geq 0$$
  $$\boxed{D_f = \langle -3, 3 \rangle}$$
- $g(x) = \dfrac{x+1}{x^2 - 1}$
  The formula doesn't make sense if the denominator is zero, so
  $$x^2 - 1 \neq 0 \quad \Leftrightarrow \quad x \neq \pm 1$$
  $$\boxed{D_g = \mathbb{R} \setminus \{-1, 1\}}$$
- $h(x) = \dfrac{3 - x}{\sqrt{x^2 - 4}}$
  This expression makes sense if the denominator is nonzero and the expression under the square root is nonnegative, so together we have
  $$x^2 - 4 > 0$$
  $$\boxed{D_h = \{x \in \mathbb{R} \;:\; x < -2 \;\vee\; x > 2\} = (-\infty, -2) \cup (2, \infty)}$$

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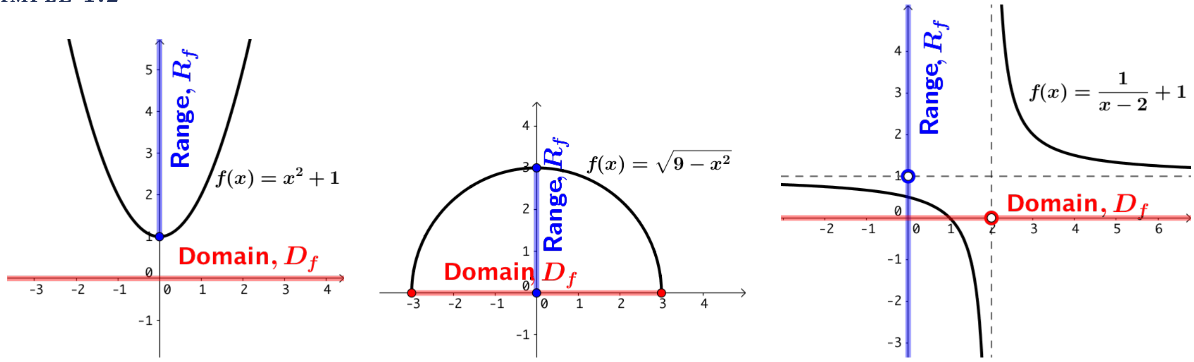
Def. 1.2 The **graph of the function** f is the set of all points (x, y) in the $xy-$plane (Cartesian plane) that satisfy the equation $f(x) = y$ for $x \in X$,

$$\{(x, y) \in \mathbb{R}^2 \;:\; x \in X, y = f(x)\}$$

- The domain D_f is the projection of the graph onto the $x-$axis (the primary axis)
- The range R_f is the projection of the graph onto the $y-$axis (the secondary axis)

Example 1.2



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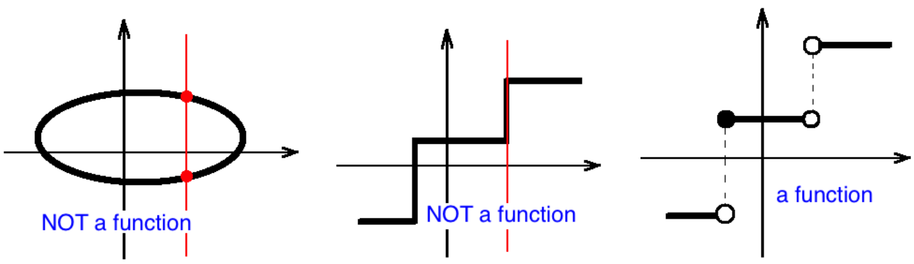
The requirement that a function must assign a *unique* value of the dependent variable to each value in the domain is expressed in the **vertical line test**.

Vertical line test.
A graph represents a function if and only if every vertical line $x = x_0$ intersects the graph at most once.



Def. 1.3 The **zero (root) of the function** f is an argument x_0 such that $f(x_0) = 0$. It is the point of intersection of the graph and the $x-$axis.

Example 1.3 Let's find the zeros of the functions given in the previous example.

- $f(x) = x^2 + 1$
  This function has no roots (as you can see on the graph), since $x^2 + 1 = 0$ has no solution.
- $f(x) = \sqrt{9 - x^2}$
  $$\sqrt{9 - x^2} = 0 \quad \Leftrightarrow \quad 9 - x^2 = 0 \quad \Leftrightarrow \quad x = \pm 3$$
  This function has two zeros, $\boxed{x = 3 \ \vee \ x = -3}$.
- $f(x) = \dfrac{1}{x-2} + 1$
  To determine the zeros of the function we equate it to zero,
  $$\frac{1}{x-2} + 1 = 0 \quad \Leftrightarrow \quad \frac{1}{x-2} = -1 \quad \Leftrightarrow \quad x - 2 = -1 \quad \Leftrightarrow \quad \boxed{x = 1}$$

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2 Operations on functions

In order to discuss operations on functions we must first understand what it means that two functions are equal.

Def. 2.1 Two functions $f : D_f \to Y_f$ and $g : D_g \to Y_g$ are equal, which we denote by $f = g$, if and only if their domains are identical and their values are the same for all x in the domain,

$$D_f = D_g \quad \wedge \quad \forall_{x \in D_f}\ f(x) = g(x)$$

Example 2.1 Determine whether the given functions are equal.

- $f(x) = x, \quad g(x) = \sqrt{x^2}$. Start with the domains,

  $$D_f = \mathbb{R}, \quad D_g = \mathbb{R}$$

  Domains are equal, but since $\sqrt{x^2} = |x|$ the functions will not have the same values for all $x \in D_f$, e.g.

  $$f(-2) = -2 \neq g(-2) = \sqrt{(-2)^2} = 2$$

  So, the functions are not equal.

- $f(x) = \sqrt{x-1}\sqrt{x-2}, \quad g(x) = \sqrt{(x-1)(x-2)}$. First we find the domains,

  $$D_f \ : \begin{cases} x - 1 \geq 0 \\ x - 2 \geq 0 \end{cases} \quad \Rightarrow \quad x \geq 2$$

  $$D_g \ : \ (x-1)(x-2) \geq 0 \quad \Rightarrow \quad x \in (-\infty, 1\rangle \cup \langle 2, \infty)$$

  Since clearly $D_f \neq D_g$, the two functions are not equal. Notice, however, that if we restrict the domain of g to the set $D = \langle 2, \infty)$, then the two functions will be equal, since $\sqrt{x-1}\sqrt{x-2} = \sqrt{(x-1)(x-2)}$.

- $f(x) = \dfrac{1}{x+2}, \quad g(x) = \dfrac{x-2}{x^2-4}$. The domains of the functions are

  $$D_f = \mathbb{R} \setminus \{-2\}, \qquad D_g = \mathbb{R} \setminus \{-2.2\}$$

  So the functions are not equal. Notice that

  $$\frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2},$$

  so the functions have the same values except for at $x = -2$. Thus, if we consider the functions on the set $D = D_g = \mathbb{R} \setminus \{-2, 2\}$ then the functions will be equal.

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Def. 2.2 If f and g are functions with domains D_f and D_g, respectively, we define

- the **sum** of f and g
  $$(f+g)(x) = f(x) + g(x), \quad D_{f+g} = D_f \cap D_g$$
- the **difference** of f and g
  $$(f-g)(x) = f(x) - g(x), \quad D_{f-g} = D_f \cap D_g$$
- the **product** of f and g
  $$(f \cdot g)(x) = f(x) \cdot g(x), \quad D_{f \cdot g} = D_f \cap D_g$$
- the **quotient** of f and g
  $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad D_{f/g} = \{x \in \mathbb{R} \ : \ x \in D_f \cap D_g \ \wedge \ g(x) \neq 0\}$$

Example 2.2 Let $f(x) = x + \sqrt{x-2}$ and $g(x) = 1 - x$. Their domains are $D_f = \langle 2, \infty)$ and $D_g = \mathbb{R}$, and the sum of these functions is

$$(f+g)(x) = x + \sqrt{x-2} + 1 - x = 1 + \sqrt{x-2}$$

The domain of the sum is

$$D_{f+g} \,:\, x \geq 2 \,\wedge\, x \in \mathbb{R}$$
$$D_{f+g} = \langle 2, \infty)$$

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Example 2.3 Let $f(x) = \sqrt{4x}$ and $g(x) = \sqrt{x}$.
The domains of both functions are the same, i.e. $D_f = D_g = \langle 0, \infty)$.

- The product of the functions is
$$(f \cdot g)(x) = \left(\sqrt{4x}\right) \cdot \left(\sqrt{x}\right) = 2x$$
Even though the natural domain of $2x$ is $\mathbb{R}$, the domain of the product is the intersection of D_f and D_g, which in this case is
$$D_{f \cdot g} = \langle 0, \infty)$$
- The quotient of the functions is
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{4x}}{\sqrt{x}} = 2$$
Again, even though the natural domain of the constant function 2 is $\mathbb{R}$, the domain of the quotient is the intersection of D_f and D_g, minus the arguments for which $g(x) = 0$, that is
$$D_{f/g} = (0, \infty)$$

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Def. 2.3 Given two functions f and g, the **composite function** $f \circ g$ (or **composition** of f and g) is defined by

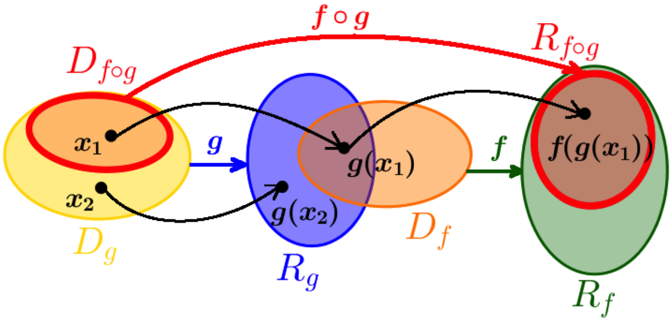
$$(f \circ g)(x) = f(g(x))$$

Domain of $f \circ g$ consists of all x in the domain of g such that $g(x)$ is in the domain of f.

$$D_{f \circ g} = \{x \in \mathbb{R} \,:\, x \in D_g \,\wedge\, g(x) \in D_f\}$$

The function g is called the **inner** function, and f is called the **outer** function.

Composition of functions is evaluated in two steps, for the given x we first evaluate the value of the inner function, and then substitute the obtained value for the argument in the outer function. The domain of the composite function is a subset of the domain of the inner function, and the range of the composition is a subset of the range of the outer function.



Example 2.4 Given $f(x) = \dfrac{1}{x}$ and $g(x) = x - 2$, find $f \circ g$, $g \circ f$, $f \circ f$ and their domains.

$$D_f = \mathbb{R} \setminus \{0\}, \quad D_g = \mathbb{R}$$

- $(f \circ g)(x) = f(g(x)) = f(x-2) = \dfrac{1}{x-2}$
  $D_{f \circ g} = \{x \in D_g \ \wedge \ (x-2) \in D_f\} = \{x \in \mathbb{R} \ \wedge \ x - 2 \neq 0\} = \mathbb{R} \setminus \{2\}$
- $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 2$
  $D_{g \circ f} = \{x \in D_f \ \wedge \ \dfrac{1}{x} \in D_g\} = \mathbb{R} \setminus \{0\}$
- $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{\frac{1}{x}} = x$
  $D_{f \circ f} = \{x \in D_f \ \wedge \ \dfrac{1}{x} \in D_f\} = \{x \neq 0 \ \wedge \ \dfrac{1}{x} \neq 0\} = \mathbb{R} \setminus \{0\}$

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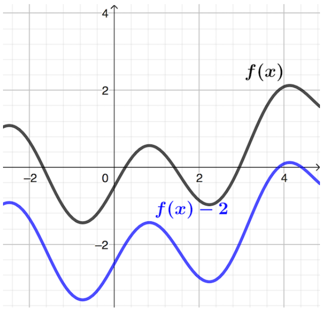
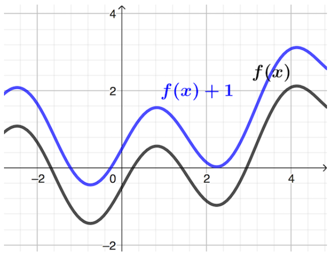
3 Graph transformations

There are several ways in which the graph of a function can be transformed to produce graphs of new functions. The common transformations are shifts, scalings, and reflections, which can be done in both $x-$ and $y-$directions. These transformations are summarized below.

- Vertical shifting

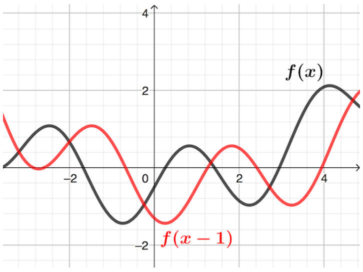
  The graph of $y = f(x) + q$ is the graph of $y = f(x)$ shifted vertically by q units (up if $q > 0$ and down if $q < 0$).

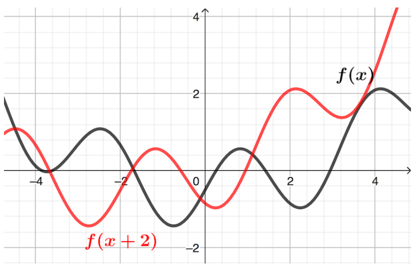




- Horizontal shifting

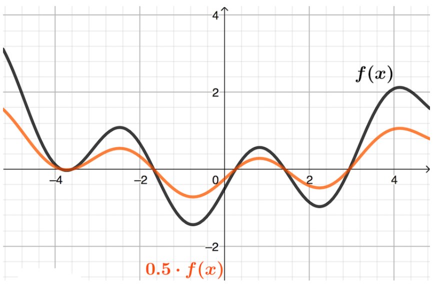
  The graph of $y = f(x - p)$ is the graph of $y = f(x)$ shifted horizontally by p units (right if $p > 0$ and left if $p < 0$).

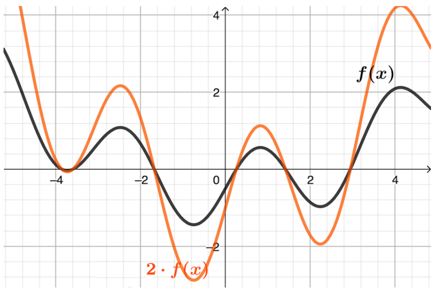




- Vertical scaling

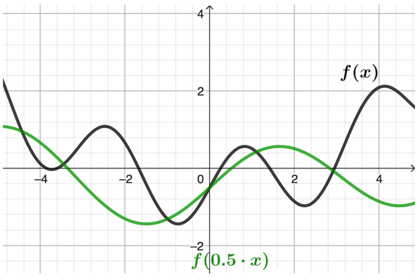
  For $c > 0$, the graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ scaled vertically by a factor of c (squeezed if $0 < c < 1$ and stretched if $c > 1$).

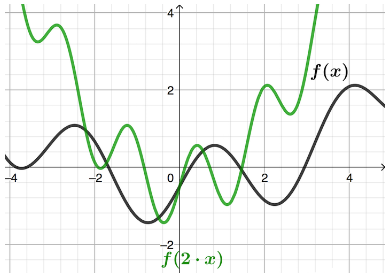




- Horizontal scaling

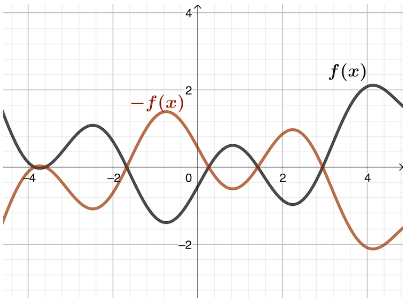
  For $a > 0$, the graph of $y = f(a \cdot x)$ is the graph of $y = f(x)$ scaled horizontally by a factor of a (broadened if $0 < a < 1$ and narrowed if $a > 1$).

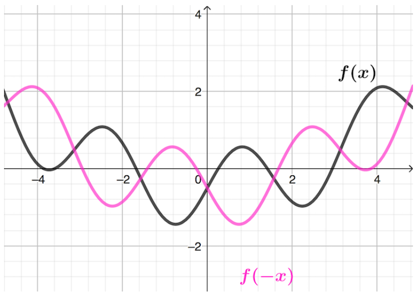




- Reflections

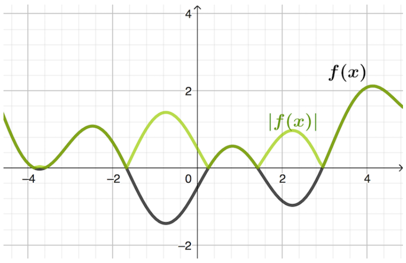
  The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the $x-$axis.
  The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the $y-$axis.

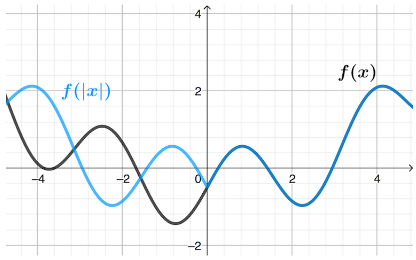




- Absolute value

  The graph of $y = |f(x)|$ is the graph of $y = f(x)$ with the parts below the $x-$axis reflected about it.
  The graph of $y = f(|x|)$ is the graph of $y = f(x)$ with its left-hand side part ignored and its right-hand side reflected about the $y-$axis.





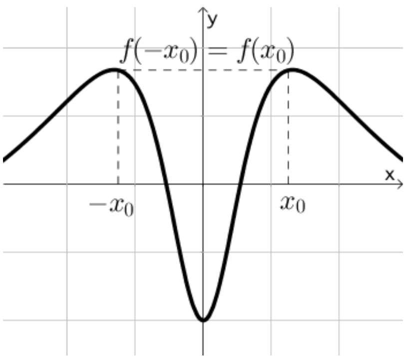
4 Properties of functions

Def. 4.1
The function $f : X \to Y$ is called **even** if for every $x \in X$

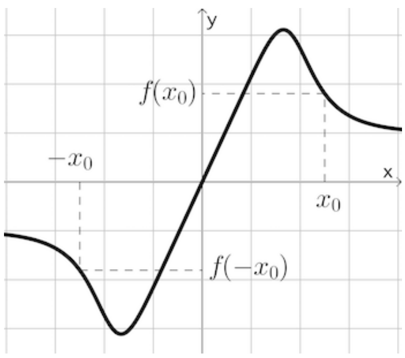
$$f(-x) = f(x) \quad \wedge \; -x \in X$$

The function $f : X \to Y$ is called **odd** if for every $x \in X$

$$f(-x) = -f(x) \quad \wedge \; -x \in X$$



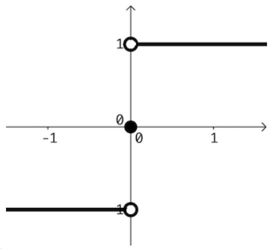
The graph of an even function is symmetric about the $y-$axis.



The graph of an odd function is symmetric about the origin.

Example 4.1 An interesting example of an odd function is the signum (sign) function that extracts the sign of a real number.

$$f(x) = \operatorname{sgn} x = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases}$$



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Example 4.2 Determine whether the given function is even, odd, or neither

- $f(x) = \dfrac{x}{x^3 + x^5} + x^2$

  As usual the domain goes first,

$$D_f \;:\; x^3 + x^5 \neq 0 \quad \Leftrightarrow \quad x^3(1 + x^2) \neq 0 \quad \Leftrightarrow \quad x^3 \neq 0$$

$$D_f = \mathbb{R} \setminus \{0\}$$

  Then we check the symmetry by definition,

$$f(-x) = \frac{-x}{(-x)^3 + (-x)^5} + (-x)^2 = \frac{-x}{-x^3 - x^5} + x^2 = \frac{x}{x^3 + x^5} + x^2 = f(x)$$

  So, the function is even.

- $g(x) = (x^2 + 1) \cdot \operatorname{sgn} x, \quad D_g = \mathbb{R}$
  Instead of going by definition, let's first show a general fact that

  the product of an even and an odd function is an odd function

  Let $p(x)$ be an even function, and let $q(x)$ be an odd function, that is

$$p(-x) = p(x)$$
$$q(-x) = -q(x)$$

  Let $h(x) = p(x) \cdot q(x)$. Then,

$$h(-x) = p(-x) \cdot q(-x) = p(x) \cdot (-q(x)) = -p(x) \cdot q(x) = -h(x)$$

  Which completes the proof of the fact.
  Now back to our problem. The function $g(x) = (x^2 + 1) \cdot \operatorname{sgn} x$ is a product of an odd ($\operatorname{sgn} x$) and an even ($x^2 + 1$) function. By the above fact we conclude that $g(x)$ is an odd function.

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Def. 4.2 A function $f : X \to Y$ is called **strictly increasing (decreasing),** if for every $x_1, x_2 \in X$ such that $x_1 < x_2$, we have

$$f(x_1) < f(x_2) \quad (f(x_1) > f(x_2))$$

A function $f : X \to Y$ is called **non-decreasing (non-increasing),** if for every $x_1, x_2 \in X$ such that $x_1 < x_2$, we have

$$f(x_1) \leq f(x_2) \quad (f(x_1) \geq f(x_2))$$

Functions with these properties are called **monotonic**.

Def. 4.3 A function $f : X \to Y$ is called **periodic** if there exists a real number $P \neq 0$ such that for any $x \in X$

$$x + P \in X \quad \wedge \quad f(x+P) = f(x)$$

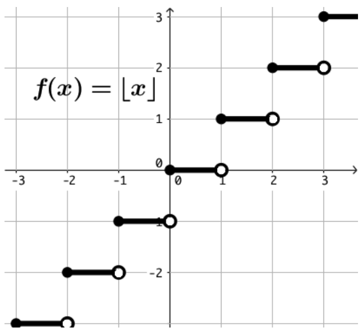
The number P is called a **period**. The smallest positive period, if it exists, is called the **fundamental period**.

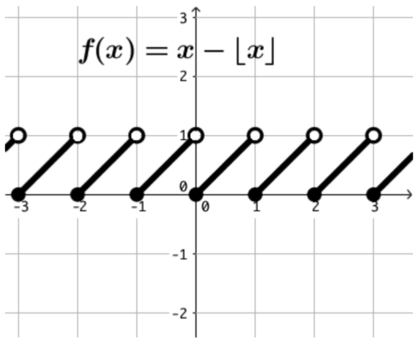
Note: not every function has a fundamental period. For example, a constant function $f(x) = c$ defined on the entire real line is a periodic function, and its period P is any non-zero real number since $f(x+P) = f(x)$ for any $x \in I\!R$ and $P \neq 0$. However, this function doesn't have a fundamental period because there is no smallest real number.

Example 4.3 Typical examples of periodic functions are trigonometric functions. Here is a less typical example. The **floor function**, $\lfloor x \rfloor$, gives the largest integer less than or equal to x

$$\lfloor x \rfloor = k \quad \Longleftrightarrow \quad k \leq x < k+1, \quad k \in \mathbb{Z}$$

The **mantissa**, $\lfloor x \rfloor - x$, is defined as the positive fractional part of x.





The mantissa function $f(x) = x - \lfloor x \rfloor$ is a periodic function, which is clear from its graph and may be confirmed by the following argument.

$$\text{Let } \lfloor x \rfloor = m \quad \Rightarrow \quad m \leq x < m+1 \quad \Rightarrow \quad m+k \leq x+k < m+k+1$$

Since $m+k$ and $m+k+1$ are consecutive integers, then

$$\lfloor x+k \rfloor = m+k = \lfloor x \rfloor + k$$

where $k \in \mathbb{Z}$. Thus,

$$f(x+k) = x+k - \lfloor x+k \rfloor = x+k - (\lfloor x \rfloor + k) = x - \lfloor x \rfloor = f(x)$$

Which proves that the function is periodic, and any integer $k \neq 0$ is its period.

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Def. 4.4 A function $f : X \to Y$ is called **one-to-one (injection)** if it never takes on the same value twice, i.e. for any $x_1, x_2 \in X$

$$x_1 \neq x_2 \quad \Longrightarrow \quad f(x_1) \neq f(x_2) \qquad \text{notation, } f : X \xrightarrow{1-1} Y$$

A function $f : X \to Y$ is called **"onto" (surjection)** if for every $y \in Y$ there exists $x \in X$ with $y = f(x)$, i.e. $R_f = Y$

$$\text{notation, } f : X \xrightarrow{onto} Y$$

A function that is both 1-1 and "onto" is called the **bijection**.

$$\text{notation, } f : X \xrightarrow[onto]{1-1} Y$$

Example 4.4 Classical example of a function that is not one-to-one is $f(x) = x^2$. There are of course others,

- $f(x) = 4x^2 - x^3$ is not one-to-one because for instance $f(0) = 0 = f(4)$ – for two different arguments the function produces the same value.
- The signum, the floor and the mantissa functions are not one-to-one.

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Example 4.5 The "onto" property is a bit easier. Any function $f : X \to Y$ can be a surjection if only it is defined in such a way that $R_f = Y$. For example,

$$f(x) = x^2, \quad f : I\!R \to I\!R \quad \text{is not onto because } R_f = \langle 0, \infty) \neq I\!R$$

But if we define it as

$$f(x) = x^2, \quad f : I\!R \to I\!R_+ \cup \{0\} \quad \text{then it is onto.}$$

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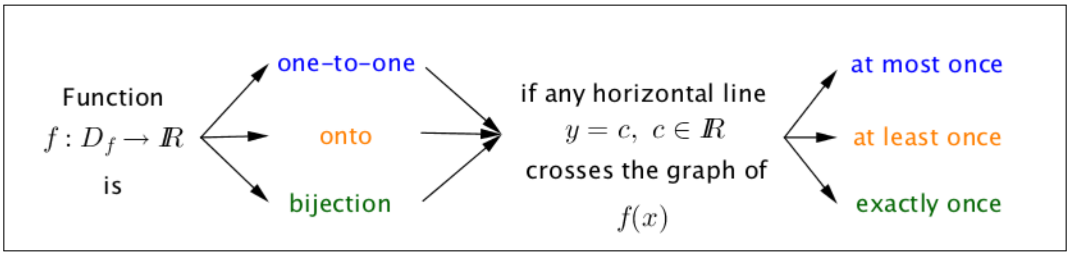
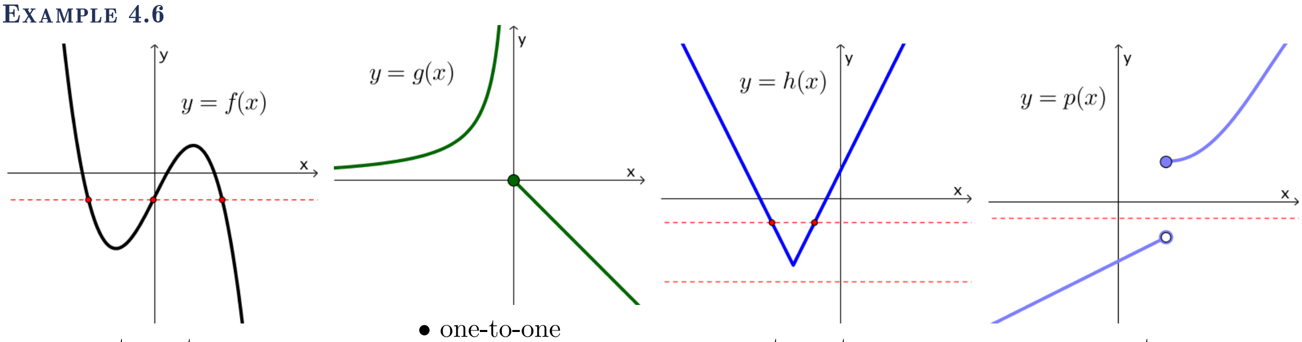
It is very easy to determine the above properties from a graph of a function. We had a vertical line test for checking whether a graph represents a function, we have a **horizontal line test** for determining whether a function is injection, surjection, or bijection.



Example 4.6



- not one-to-one
- onto

- one-to-one
- onto

$\Longrightarrow$ bijection

- not one-to-one
- not onto

- one-to-one
- not onto

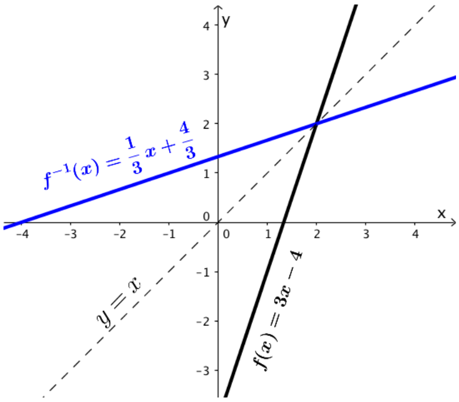
—— ✓ ✓ ✓ ——

Def. 4.5 Given a bijective function f, its **inverse function** is a function f^{-1} such that whenever $y = f(x)$ then $f^{-1}(y) = x$, and the domain of f is the range of f^{-1}, and the domain of f^{-1} is the range of f.

$$f : X \to Y, \quad f^{-1} : Y \to X, \qquad f^{-1}(y) = x \quad \Longleftrightarrow \quad y = f(x)$$

The graphs of f and f^{-1} are symmetric about the line $y = x$.

Example 4.7 Let's find inverse functions of the ones given below

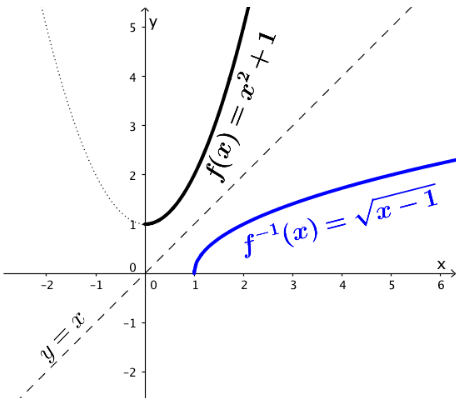


- $f(x) = 3x - 4, \quad x \in I\!R$

  This is an increasing linear function with $D_f = I\!R$ and $R_f = I\!R$, and it is a bijection. Therefore f^{-1} exists, and we may find it by rearranging the equation

  $$y = 3x - 4 \quad \Leftrightarrow \quad 3x = y + 4 \quad \Leftrightarrow \quad x = \frac{1}{3}\, y + \frac{4}{3}$$

  So, $\quad f^{-1}(x) = \frac{1}{3}\, x + \frac{4}{3}$ with $D_{f^{-1}} = I\!R,\ R_{f^{-1}} = I\!R$



- $f(x) = x^2 + 1, \quad x \geq 0$

  Notice that if the domain of the function was not restricted but the entire real line instead, then it would not be a bijection (for instance, $f(-1) = f(1) = 2$) and so the inverse would not exist. If we, however, restrict the domain to $D_f = \langle 0, \infty)$ then $R_f = \langle 1, \infty)$ and the function is a bijection ($f : D_f \to R_f$). We can find the formula the same way as before,

  $$y = x^2 + 1 \quad \Leftrightarrow \quad x^2 = y - 1 \quad \Leftrightarrow \quad x = \sqrt{y - 1}$$

  So, $\quad f^{-1}(x) = \sqrt{x - 1}$ with $D_{f^{-1}} = \langle 1, \infty)\,,\ R_{f^{-1}} = \langle 0, \infty)$

—— ✓ ✓ ✓ ——

References

[1] *Matematyka – podstawy z elementami matematyki wyszej*, edited by B. Wikieł , PG publishing house, 2009.