The word *series* as used in mathematics is different from the way it is used in everyday speech. *Webster's Third New International Dictionary* includes the following two distinct meanings:

1. a group of usually three or more things or events standing or succeeding in order and having a like relationship to each other; a spatial or temporal succession of persons or things.
2. the expression obtained from a mathematical sequence by connecting its terms with plus signs.

For our purposes, we will consider the second of these definitions.

EXAMPLE

We know how to add a finite collection of numbers. But suppose we wish to add the infinite collection of numbers

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.$$

The way to do so is not to try to add all the terms at ones (we can't) but rather to add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

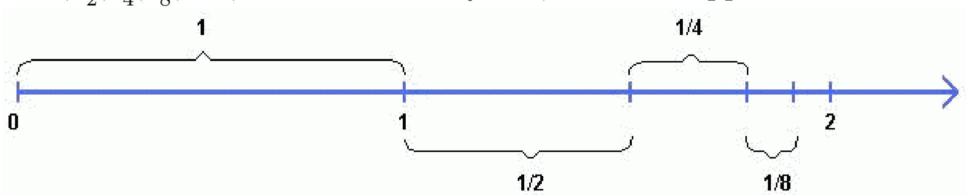
Partial sum	Value
$s_1 = 1$	1
$s_2 = 1 + \frac{1}{2}$	$2 - \frac{1}{2}$
$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$2 - \frac{1}{4} = 2 - \frac{1}{2^2}$
$s_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$2 - \frac{1}{8} = 2 - \frac{1}{2^3}$
$\ldots$	$\ldots$
$s_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$

There is a pattern. The partial sums form a sequence whose nth term is

$$s_n = 2 - \frac{1}{2^{n-1}} \ ,$$

and this sequence converges to 2. We therefore say: "the sum of the infinite series is 2".

As the lenghts 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ..., are added one by one, the sums approach 2:



In general - finding a pattern for a partial sums of a series is difficult, often impossible.

DEFINITION

Given a sequence of numbers (a_n), an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is called an infinite series. The number a_n is called the nth term of the series. The sequence (s_n) defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\ldots$$

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

is the sequence of partial sums of the series, the number s_n being the nth partial sum.
If the sequence of partial sums converges to a limit L, we say that the series converges and that its sum is L. In this case we also write

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = L \,.$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

NOTATIONS

$$\sum_{n=1}^{\infty} a_n \,, \quad \sum a_n$$

The first of these is read *summation, from* n *equals* 1 *to infinity, of terms* a_n, the second as *summation* a_n.

EXAMPLE

Show that the series $\sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$ diverges.

EXAMPLE

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, and find its sum.

A DIVERGENCE TEST

Our first theorem about series will tell us immediately that certain series diverge.

THEOREM

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
- If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist), then $\sum_{n=1}^{\infty} a_n$ diverges.

EXAMPLE

Show that the series $\sum_{n=1}^{\infty} \cos\frac{1}{n}$ diverges.

EXAMPLE

Show that the series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

THE HARMONIC SERIES AND THE p-SERIES

THEOREM

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

diverges.

THEOREM

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots,$$

where p is a real constant

- converges if $p > 1$,
- diverges if $p \leq 1$.

GEOMETRIC SERIES

Geometric series are series of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real number and $a \neq 0$. The ratio r can be positive or negative.

Theorem

- If $|r| < 1$, the geometric series converges, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \, .$$

- If $|r| \geq 1$, the geometric series diverges (unless $a = 0$).

Example

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a \cdot 1 + a \cdot 1^2 + \ldots + a \cdot 1^{n-1} = na \, ,$$

and the series diverges becouse $\lim\limits_{n \to \infty} s_n = \pm\infty$.

Theorem

For $\sum\limits_{n=m}^{\infty} px^n$, when $p \neq 0$ and $|x| < 1$, we have

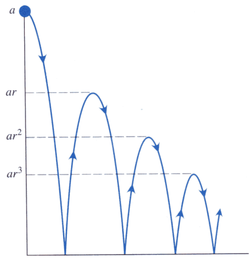
$$\sum_{n=m}^{\infty} px^n = \frac{px^m}{1-x} \, .$$

Example

Show that: a) $\sum\limits_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4}{3}$; b) $\sum\limits_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{8}{3}$.

Example

A ball is dropped from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is a positive number less than 1. Find the total distance the ball travels up and down.



Whenever we have two convergent series, we can add them, substract them, and multiply them by constans to make other cenvergent series.

Theorem
If $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, then

- Sum rule: $\sum_{n=0}^{\infty}(a_n + b_n) = A + B$
- Difference rule: $\sum_{n=0}^{\infty}(a_n - b_n) = A - B$
- Constant Multiple Rule: $\sum_{n=0}^{\infty} ka_n = kA$ (any number k)

Theorem
A nondecreasing sequence converges if and only if its terms are bounded from above. If all terms are less then or equal to M, then the limit of the sequence is less then or equal to M as well.

This theorem tells us that we can show that a series $\sum a_n$ of nonnegative terms converges if we can show that its prtial sums are bounded from above.

Example
Show that the partial sums of $\sum_{n=0}^{\infty} \frac{1}{n!}$ are all less than 3 (it means the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges - but not necessarily to 3).

Nonnegative series - some tests for convergence

The integral test involves comparing a nonnegative series with an improper integral:

Theorem (Integral Test)
Let (a_n) be a nonnegative sequence, and let f be a continuous, decreasing function defined on $< 1, \infty)$ such that

$$f(n) = a_n$$

for $n > 1$. Then series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges.

The comparison test involves the comparison of two series:

THEOREM (Comparison Test)

- If $\sum_{n=1}^{\infty} b_n$ converges and $0 \le a_n \le b_n$ for all $n \ge 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

- If $\sum_{n=1}^{\infty} b_n$ diverges and $0 \le b_n \le a_n$ for all $n \ge 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

THEOREM (Limit comparision test)

- <u>Test for Convergence</u>. If $a_n \ge 0$ for $n > n_0$ and there is a convergent series $\sum_{n=1}^{\infty} c_n$ such that $c_n \ge 0$ and $\sum_{n=1}^{\infty} \frac{a_n}{c_n} < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges.

- <u>Test for Divergence</u>. If $a_n \ge 0$ for $n > n_0$ and there is a divergent series $\sum_{n=1}^{\infty} d_n$ such that $d_n \ge 0$ and $\sum_{n=1}^{\infty} \frac{a_n}{d_n} > 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

SIMPLIFIED LIMIT COMPARISION TEST

If the terms of two series $\sum a_n$ and $\sum b_n$ are positive for $n > n_0$ and the limit of $\frac{a_n}{b_n}$ is finite and positive, then both series converge or both diverge.

THEOREM (Ratio Test)

Assume that $a_n \neq 0$ for all n and that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r\ .$$

- If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

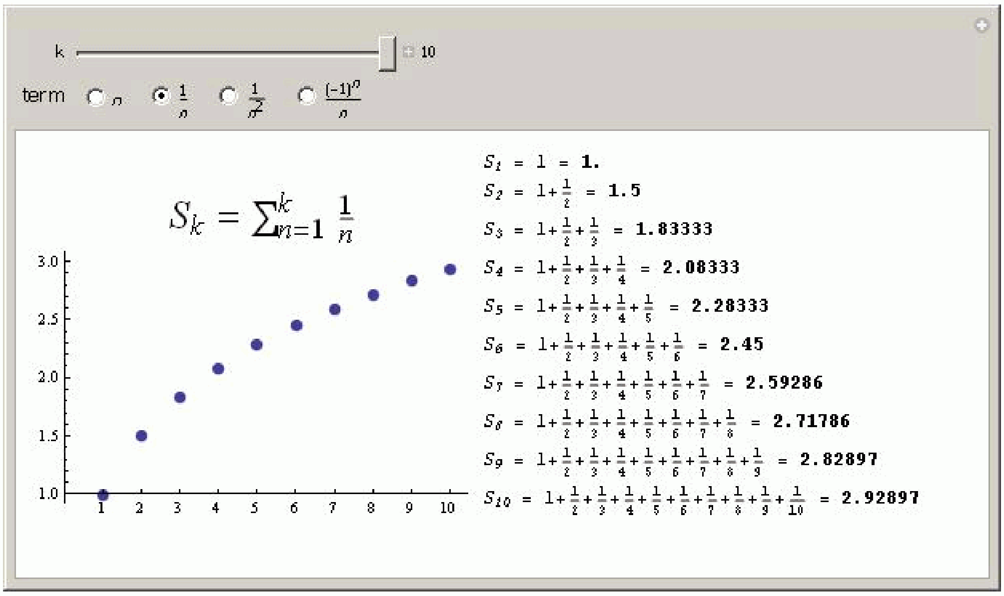
- If $r = 1$, then from this test we can not draw any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$.

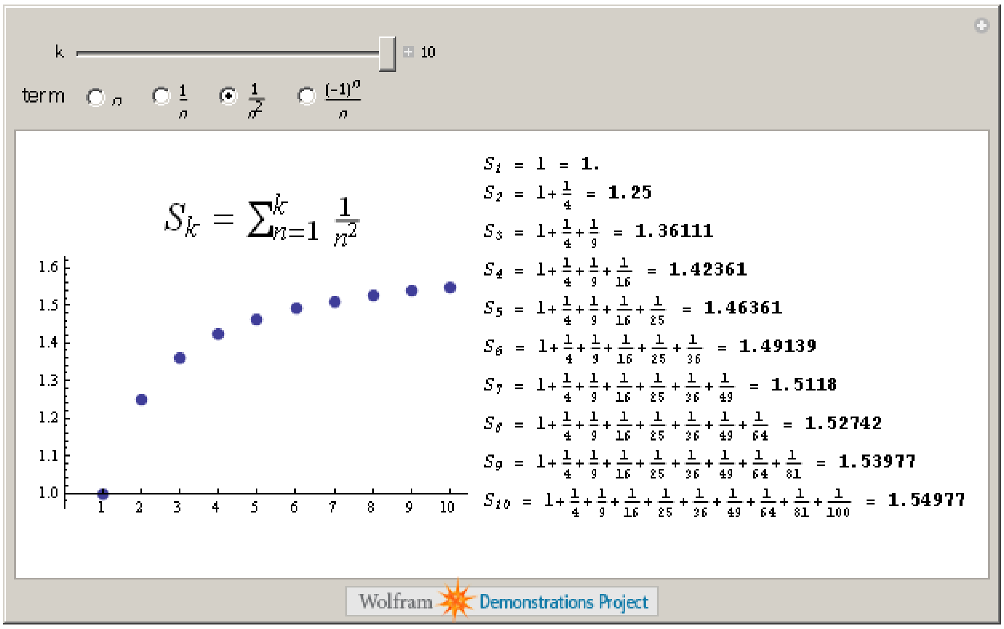
THEOREM (Root Test)

Assume that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = r\,.$$

- If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then from this test we can not draw any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$.





Example

Which series converge, and which diverge? Give reasons for your answer.

Alternating Series and Absolute Convergence

A series in which the terms are alternately positive and negative is called an <u>alternating series</u>.

Example

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ - is called the alternating harmonic series.

b) $\sum_{n=1}^{\infty} (-1)^n$

c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$

Theorem (The Alternating Series Test)

The alternating series

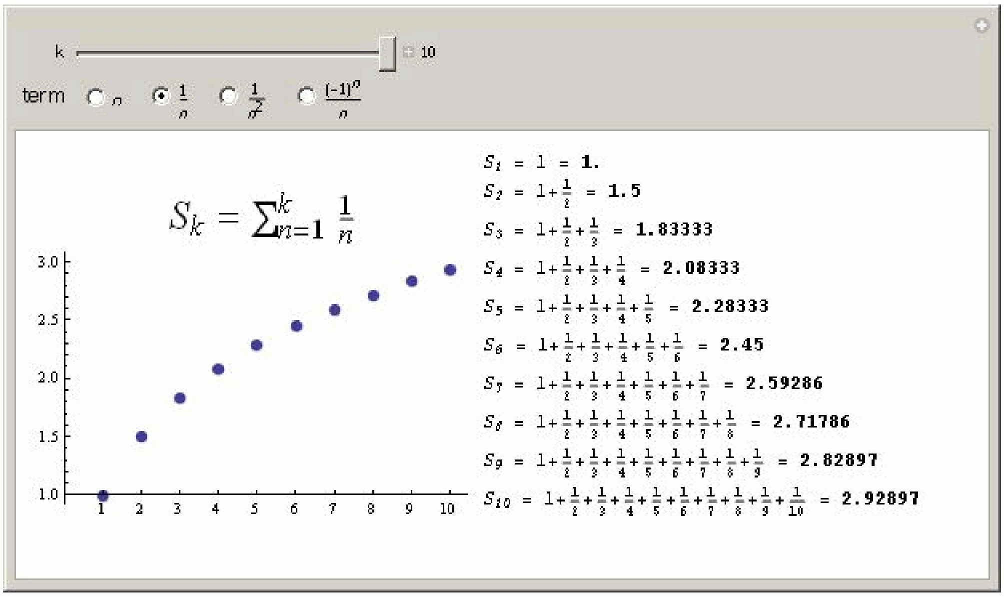
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

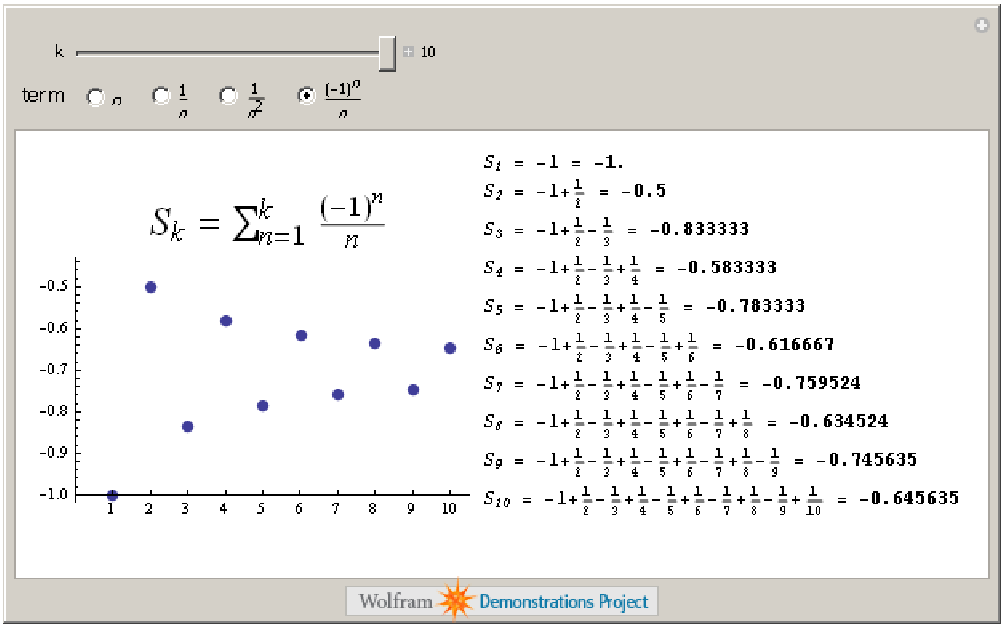
converges if all three following conditions are satisfied

- The a_n's are all positive.
- $a_n > a_{n+1}$ for all n.
- $\lim_{n\to\infty} a_n = 0$.

Example

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ satisfies the three requirments of the theorem, therefore, it converges.





ABSOLUTE CONVERGENCE

DEFINITION

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values, $\sum_{n=1}^{\infty} |a_n|$, is convergent.

A series that converges but does not converge absolutely converges conditionally.

THEOREM

If $\sum_{n=1}^{\infty} |a_n|$ coverges, then $\sum_{n=1}^{\infty} a_n$ converges.