

SUBSTITUTION I .. $f(ax + b)$

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A Tutorial Module for practising the integration of expressions of the form $f(ax + b)$

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1. Theory

Consider an integral of the form

$$\int f(ax + b)dx$$

where a and b are constants. We have here an unspecified function f of a linear function of x

Letting $u = ax + b$ then $\frac{du}{dx} = a$, and this gives $dx = \frac{du}{a}$

This allows us to change the integration variable from x to u

$$\int f(ax + b)dx = \int f(u)\frac{du}{a}$$

The final result is

$$\int f(ax + b)dx = \frac{1}{a} \int f(u) du$$

where $u = ax + b$

This is a general result for integrating functions of a linear function of x

Each application of this result involves **dividing by the coefficient of x** and then integrating

2. Exercises

Click on [EXERCISE](#) links for full worked solutions (10 exercises in total).

Perform the following integrations:

[EXERCISE 1.](#)

$$\int (2x - 1)^3 dx$$

[EXERCISE 2.](#)

$$\int \cos(3x + 5) dx$$

[EXERCISE 3.](#)

$$\int e^{5x+2} dx$$

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EXERCISE 4.

$$\int \sinh 3x \, dx$$

EXERCISE 5.

$$\int \frac{dx}{2x-1}$$

EXERCISE 6.

$$\int \frac{dx}{1+(5x)^2}$$

EXERCISE 7.

$$\int \sec^2(7x+1) \, dx$$

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EXERCISE 8.

$$\int \sin(3x - 1) dx$$

EXERCISE 9.

$$\int \cosh(1 + 2x) dx$$

EXERCISE 10.

$$\int \tan(9x - 1) dx$$

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3. Answers

1. $\frac{1}{8}(2x - 1)^4 + C$,
2. $\frac{1}{3} \sin(3x + 5) + C$,
3. $\frac{1}{5} e^{5x+2} + C$,
4. $\frac{1}{3} \cosh 3x + C$,
5. $\frac{1}{2} \ln |2x - 1| + C$,
6. $\frac{1}{5} \tan^{-1} 5x + C$,
7. $\frac{1}{7} \tan(7x + 1) + C$,
8. $-\frac{1}{3} \cos(3x - 1) + C$,
9. $\frac{1}{2} \sinh(1 + 2x) + C$,
10. $-\frac{1}{9} \ln |\cos(9x - 1)| + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right $ ($0 < x < a$) $\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $ ($ x > a > 0$)
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left \frac{x+\sqrt{a^2+x^2}}{a} \right $ ($a > 0$) $\ln \left \frac{x+\sqrt{x^2-a^2}}{a} \right $ ($x > a > 0$)
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[\sin^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[\sinh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[-\cosh^{-1} \left(\frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

5. Tips

- STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to

- When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the [Back](#) button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int (2x - 1)^3 dx$$

Let $u = 2x - 1$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

$$\begin{aligned}\therefore \int (2x - 1)^3 dx &= \int u^3 \frac{du}{2} = \frac{1}{2} \int u^3 du \\ &= \frac{1}{2} \frac{1}{4} u^4 + C \\ &= \frac{1}{8} (2x - 1)^4 + C.\end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 2$ and $\int f(u) du$ is $\int u^3 du$.

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Exercise 2.

$$\int \cos(3x + 5) dx$$

Let $u = 3x + 5$ then $\frac{du}{dx} = 3$ and $dx = \frac{du}{3}$

$$\begin{aligned} \therefore \int \cos(3x + 5) dx &= \int \cos u \cdot \frac{du}{3} = \frac{1}{3} \int \cos u du \\ &= \frac{1}{3} \sin u + C \\ &= \frac{1}{3} \sin(3x + 5) + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 3$ and $\int f(u) du$ is $\int \cos u du$.

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Exercise 3.

$$\int e^{5x+2} dx$$

Let $u = 5x + 2$ then $\frac{du}{dx} = 5$ and $dx = \frac{du}{5}$

$$\begin{aligned} \therefore \int e^{5x+2} dx &= \int e^u \frac{du}{5} = \frac{1}{5} \int e^u du \\ &= \frac{1}{5} e^u + C \\ &= \frac{1}{5} e^{5x+2} + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 5$ and $\int f(u) du$ is $\int e^u du$.

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Exercise 4.

$$\int \sinh 3x \, dx$$

Let $u = 3x$ then $\frac{du}{dx} = 3$ and $dx = \frac{du}{3}$

$$\begin{aligned} \therefore \int \sinh 3x \, dx &= \int \sinh u \cdot \frac{du}{3} = \frac{1}{3} \int \sinh u \, du \\ &= \frac{1}{3} \cosh u + C \\ &= \frac{1}{3} \cosh 3x + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) \, dx = \frac{1}{a} \int f(u) \, du$$

where $a = 3$ and $\int f(u) \, du$ is $\int \sinh u \, du$.

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Exercise 5.

$$\int \frac{dx}{2x-1}$$

Let $u = 2x - 1$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

$$\begin{aligned}\therefore \int \frac{dx}{2x-1} &= \int \frac{du}{u} \cdot \frac{1}{2} = \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln |2x-1| + C.\end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax+b) dx = \frac{1}{a} \int f(u) du$$

where $a = 2$ and $\int f(u) du$ is $\int \frac{du}{u}$.

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Exercise 6.

$$\int \frac{dx}{1 + (5x)^2}$$

Let $u = 5x$ then $\frac{du}{dx} = 5$ and $dx = \frac{du}{5}$

$$\begin{aligned} \therefore \int \frac{dx}{1 + (5x)^2} &= \int \frac{du}{1 + u^2} \cdot \frac{1}{5} = \frac{1}{5} \int \frac{du}{1 + u^2} \\ &= \frac{1}{5} \tan^{-1} u + C \\ &= \frac{1}{5} \tan^{-1} 5x + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 5$ and $\int f(u) du$ is $\int \frac{du}{1+u^2} du$.

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Exercise 7.

$$\int \sec^2(7x + 1) dx$$

Let $u = 7x + 1$ then $\frac{du}{dx} = 7$ and $dx = \frac{du}{7}$

$$\begin{aligned} \therefore \int \sec^2(7x + 1) dx &= \frac{1}{7} \int \sec^2 u du \\ &= \frac{1}{7} \tan u + C \\ &= \frac{1}{7} \tan(7x + 1) + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 7$ and $\int f(u) du$ is $\int \sec^2 u du$.

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Exercise 8.

$$\int \sin(3x - 1) dx$$

Let $u = 3x - 1$ then $\frac{du}{dx} = 3$ and $dx = \frac{du}{3}$

$$\begin{aligned} \therefore \int \sin(3x - 1) dx &= \frac{1}{3} \int \sin u \, du \\ &= -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(3x - 1) + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 3$ and $\int f(u) du$ is $\int \sin u du$.

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Exercise 9.

$$\int \cosh(1 + 2x) dx$$

Let $u = 1 + 2x$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

$$\begin{aligned} \therefore \int \cosh(1 + 2x) dx &= \int \cosh u \frac{du}{2} \\ &= \frac{1}{2} \sinh u + C \\ &= \frac{1}{2} \sinh(1 + 2x) + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 2$ and $\int f(u) du$ is $\int \cosh u du$.

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Exercise 10.

$$\int \tan(9x - 1) dx$$

Let $u = 9x - 1$ then $\frac{du}{dx} = 9$ and $dx = \frac{du}{9}$

$$\begin{aligned} \therefore \int \tan(9x - 1) dx &= \int \tan u \cdot \frac{du}{9} = \frac{1}{9} \int \tan u du \\ &= -\frac{1}{9} \ln |\cos u| + C \\ &= -\frac{1}{9} \ln |\cos(9x - 1)| + C. \end{aligned}$$

Note. The final result can also be obtained using the general pattern:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

where $a = 9$ and $\int f(u) du$ is $\int \tan u du$.

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