1 Logarithms

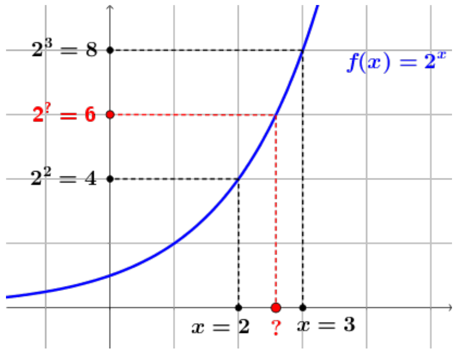
When solving exponential equations we first try to express both sides as a power with the same base so that we can drop that base,

$$2^x = 4 \quad \Leftrightarrow \quad 2^x = 2^2 \quad \Leftrightarrow \quad x = 2$$
$$2^x = 8 \quad \Leftrightarrow \quad 2^x = 2^3 \quad \Leftrightarrow \quad x = 3$$

But, what if the constant cannot be easily written as a power with the same base ?

$$2^x = 6 \quad \Leftrightarrow \quad 2^x = 2^?$$

The function $f(x) = 2^x$ is continuous, so there must be some x for which the value of 6 is attained. It's not a nice integer nor even a rational number, but it is there. The question is, how do we find it ?



The operation that determines the exponent that produces a given power is *logarithm*.

Def. 1.1 For real numbers x and a such that $x > 0, a > 0$ and $a \neq 1$ the **logarithm of x with base a** is the exponent, y, to which the base a must be raised, to produce the number x, i.e.

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

There are two special bases which give logarithms their own names and notations:

- when $a = 10$ we have a **common (decimal) logarithm**, and we simply write $\boxed{\log x}$
- when $a = e = 2,7182\ldots$ we have a **natural logarithm** which we denote by $\boxed{\ln x}$

Example 1.1

- $\log_3 81 = 4$, because $3^4 = 81$
- $\log_\pi 1 = 0$, because $\pi^0 = 1$
- $\ln e = 1$, because $e^1 = e$
- $\log_2 \frac{1}{32} = -5$, because $2^{-5} = \frac{1}{32}$
- $\log 100 = 2$, because $10^2 = 100$
- $\log_{25} \sqrt{5} = \frac{1}{4}$, because $25^{\frac{1}{4}} = 5^{\frac{1}{2}} = \sqrt{5}$

—— ✓ ✓ ✓ ——

We can draw some conclusions about logarithms directly from the definition,

Properties of logs from the definition $a > 0,\ a \neq 1,\ x \in \mathbb{R}$	
$\log_a 1 = 0$	, because $a^0 = 1$
$\log_a a = 1$	, because $a^1 = a$
$\log_a a^x = x$	
$a^{\log_a x} = x$	

Since a logarithm is the exponent of a power, we should expect the laws for logarithms to work the same as the laws for exponents. And, of course, they do,

The Logarithm Laws $a, b > 0,\ a, b \neq 1,\ x > 0,\ y > 0,\ r \in I\!R$	Corresponding Laws of Exponents $a > 0,\ a \neq 1,\ p, r \in I\!R$
$\log_a (x \cdot y) = \log_a x + \log_a y$	$a^p \cdot a^r = a^{p+r}$
for $x, y < 0$, $\log_a (x \cdot y) = \log_a (-x) + \log_a (-y)$	
$\log_a \frac{x}{y} = \log_a x - \log_a y$	$\frac{a^p}{a^r} = a^{p-r}$
for $x, y < 0$, $\log_a \frac{x}{y} = \log_a (-x) - \log_a (-y)$	
$\log_a x^r = r \log_a x$	$(a^p)^r = a^{pr}$
for $x \in I\!R - \{0\}$, r–even, $\log_a x^r = r \log_a \lvert x \rvert$	

There are also useful formulas for changing the base of a logarithm (particularly useful when evaluating logs on a calculator)

Change of base formulas $a, b > 0, a, b \neq 1, x > 0$
$\log_a x = \dfrac{\log_b x}{\log_b a}$
$\log_a b = \dfrac{1}{\log_b a}$

Example 1.2

- $\log_8 32 = \dfrac{\log_2 32}{\log_2 8} = \dfrac{5}{3}$
- $\log_{\sqrt[3]{9}} 81\sqrt{3} = \dfrac{\log_3 81\sqrt{3}}{\log_3 \sqrt[3]{9}} = \dfrac{\log_3 \left(3^4 \cdot 3^{1/2}\right)}{\log_3 3^{2/3}} = \dfrac{\log_3 3^{9/2}}{\frac{2}{3}} = \dfrac{\frac{9}{2}}{\frac{2}{3}} = \dfrac{27}{4}$
- $81^{\frac{1}{\log_5 3}} = 81^{\log_3 5} = 3^{4 \cdot \log_3 5} = \left(3^{\log_3 5}\right)^4 = 5^4 = 625$
- $27^{\log_9 36} + 3^{\frac{4}{\log_7 9}} = \left(3^3\right)^{\log_9 36} + 3^{4 \cdot \log_9 7} = \left(3^3\right)^{\frac{\log_3 36}{\log_3 9}} + 3^{4 \cdot \frac{\log_3 7}{\log_3 9}} = \left(3^3\right)^{\frac{\log_3 6^2}{2}} + 3^{4 \cdot \frac{\log_3 7}{2}} = 3^{3 \cdot \log_3 6} + 3^{2 \cdot \log_3 7} = \left(3^{\log_3 6}\right)^3 + \left(3^{\log_3 7}\right)^2 = 6^3 + 7^2 = 265$

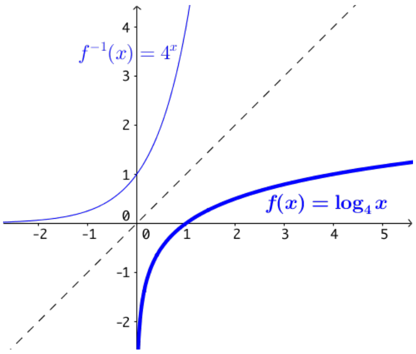
—— ✓ ✓ ✓ ——

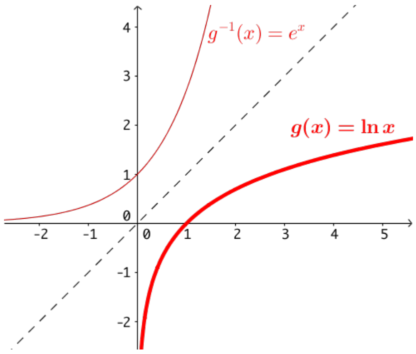
2 Logarithmic function and its properties

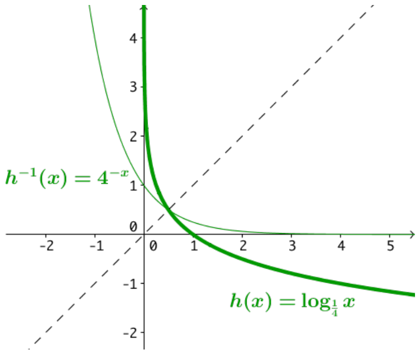
We are now ready to define the logarithmic function. It is the inverse of the exponential function, so it has the same restrictions on the base a, and its domain must be the same as the range of the exponential function, just as its range must be the same as the domain of the exponential function.

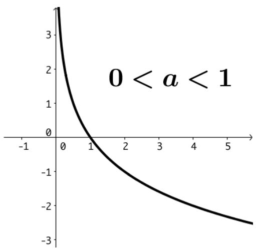
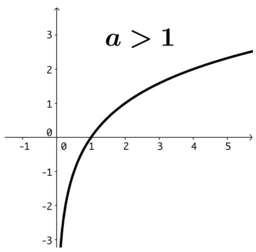
> **Def. 2.1** The function given by
> $$f(x) = \log_a x$$
> where the real numbers a and x are such that $a > 0, a \neq 1, x > 0$, is called the **logarithmic function with base a**.
> If $a = e = 2,71828\ldots$ then it is called the **natural logarithmic function** an is denoted by $\ln x$.

Since **we can read most of the properties of a function from its graph**, let's draw the graphs. Exponential function came in two varieties, so does the logarithmic function, and since they are inverses of one another, their graphs must be symmetric about the line $y = x$. Below you see several examples of graphs of logarithmic functions, followed by a table with general graph shapes and properties of log functions.







Properties of the logarithmic function $f(x) = \log_a x$	
 decreasing	 increasing
one-to-one zero: $x = 1$ domain: $\mathbb{R}_+$ range: $\mathbb{R}$	

EXAMPLE 2.1 Determine which number is bigger , $\log_3 123$ or $\log_4 234$.

We'll use the fact that the function $f(x) = \log_a x$ with $a > 1$ is increasing

$$\log_4 234 < \log_4 256 = \log_4 4^4 = 4 = \log_3 3^4 = \log_3 81 < \log_3 123$$

—— ✓ ✓ ✓ ——

Example 2.2 Determine the domain of the function $f(x) = \log_{x+0.5}(x^3 - x)$.

Since the variable x appears in both the argument and the base of the function, we will have three restrictions on the domain,

$$D_f \; : \begin{cases} x^3 - x > 0 \\ x + 0.5 > 0 \\ x + 0.5 \neq 1 \end{cases} \Longrightarrow \begin{cases} x(x-1)(x+1) > 0 \\ x > -0.5 \\ x \neq 0.5 \end{cases} \Longrightarrow \begin{cases} x \in (-1,0) \cup (1,\infty) \\ x > -0.5 \\ x \neq -0.5 \end{cases} \Longrightarrow x \in (-0.5, 0) \cup (1, \infty)$$

—— ✓ ✓ ✓ ——

Example 2.3 Determine the domain and range, and find the inverse function of $f(x) = 4 - 5^{2x-1}$.

This is an exponential function with no restrictions on the domain, so $D_f = \mathbb{R}$.

To determine the range of f we can either start with finding the inverse, and then find its domain, which will be the range we need, or we can do the following,

$$5^{2x-1} \in (0, +\infty) \;\Rightarrow\; -5^{2x-1} \in (-\infty, 0) \;\Rightarrow\; 4 - 5^{2x-1} \in (-\infty, 4)$$

Thus, the range of the function is $R_f = (-\infty, 4)$.
Now we find the inverse,

$$\begin{aligned} y &= 4 - 5^{2x-1} \\ 5^{2x-1} &= 4 - y \\ \log_5(4 - y) &= 2x - 1 \\ 2x &= \log_5(4-y) + 1 \\ x &= \frac{\log_5(4-y) + 1}{2} \end{aligned}$$

So,

$$f^{-1}(x) = \frac{1}{2}\left(\log_5(4-x) + 1\right), \quad \text{with } D_{f^{-1}} = R_f = (-\infty, 4)$$

—— ✓ ✓ ✓ ——

• Logarithmic equations

$$\boxed{\textbf{one-to-one property: } \log_a x = \log_a y \quad \Leftrightarrow \quad x = y}$$

The one-to-one property is used when solving logarithmic equations. The idea is the same as with exponential equations, we try to write both sides of the equation as logarithms with the same base, so that we can drop them.

Example 2.4 Let's solve the equation $\log_3(5x+2) - \log_3(8-x) = 2$.

We must start with restrictions on the variable x, due to the domain of the logarithm function.

$$\text{Assumptions: } \begin{cases} 5x + 2 > 0 \\ 8 - x > 0 \end{cases} \Rightarrow \begin{cases} x > -\frac{2}{5} \\ 8 > x \end{cases} \Longrightarrow x \in \left(-\frac{2}{5}, 8\right)$$

Now we simplify the equation. The left-hand side may be combined into one logarithm

$$\log_3 \frac{5x+2}{8-x} = 2$$

Here we can either use the definition of the logarithm, or change the right-hand side of the equation into the logarithm of the same base

$$\log_3 \frac{5x+2}{8-x} = \log_3 3^2$$

and then use the one-to-one property and drop the logarithms

$$\frac{5x+2}{8-x} = 9$$

This is now a rational equation, which simplifies to

$$14x = 70 \Longrightarrow \boxed{x = 5}$$

The value found from the equation satisfies the assumptions that we started with, so it is the final solution.

—— ✓ ✓ ✓ ——

Example 2.5 Logarithms may be useful when solving exponential equations. Let's take this simple example,

$$2^{x+4} = 5$$

To drop the base 2 of the exponential function we first must change the number 5 into a power with base 2, or we can use an inverse operation, which is evaluating the logarithm (just like we multiply or divide both sides of an equation by a number, we can apply other operations like evaluating logs or exponentials).

one-to-one property	or	inverse function
$2^{x+4} = 2^{\log_2 5}$		$\log_2 2^{x+4} = \log_2 5$
drop the base		simlify LHS

and we get exactly the same equation

$$x + 4 = \log_2 5$$

$$\boxed{x = \log_2 5 - 4}$$

—— ✓ ✓ ✓ ——

Example 2.6 Sometimes substitution may be useful in simplifying the equation. Consider the equation

$$\log^2 x + 3\log(10x) = 7$$

As usual, we start with assumptions (or domain),

$$\text{assumptions:} \begin{cases} x > 0 \\ 10x > 0 \end{cases} \Longrightarrow \quad x > 0$$

Since one of the logs is squared it is impossible to combine the left-hand side into one logarithm. Instead, we will use one of the properties given in a table earlier to change the equation into a quadratic one,

$$\log^2 x + 3(\log 10 + \log x) = 7$$

$$\log^2 x + 3(1 + \log x) = 7$$

$$\log^2 x + 3\log x - 4 = 0$$

We may now (though, we don't have to) make a simple substitution by letting $\log x = t$, which clearly shows us a quadratic equation

$$t^2 + 3t - 4 = 0$$

$$(t-1)(t+4) = 0$$

$$t = 1 \quad \vee \quad t = -4$$

$$\log x = 1 \quad \vee \quad \log x = -4$$

$$\boxed{x = 10 \quad \vee \quad x = 0.0001}$$

—— ✓ ✓ ✓ ——

• Logarithmic inequalities

When solving inequalities with a logarithmic function we most often use the monotonicity property.

Monotonicity

$$\log_a f(x) \le \log_a g(x) \iff 0 < f(x) \le g(x), \quad \text{for } a > 1$$
$$\log_a f(x) \le \log_a g(x) \iff f(x) \ge g(x) > 0, \quad \text{for } 0 < a < 1$$

Example 2.7 We start we a simple inequality

$$\log(x+4) - \log 21 \ge \log x$$

First, assumptions on x

$$\begin{cases} x+4>0 \\ x>0 \end{cases} \implies x > 0$$

With this, we can use properties of logs to change the left-hand side,

$$\log \frac{x+4}{21} \ge \log x$$

and by monotonicity of log with base 10, we get

$$\frac{x+4}{21} \ge x > 0$$

Since $x > 0$, we can drop the right inequality and simplify to get

$$x + 4 \ge 21x \implies x \le \frac{1}{5}$$

Thus,

$$\left(x \le \frac{1}{5}\right) \wedge (x > 0) \implies \boxed{x \in \left(0, \frac{1}{5}\right\rangle}$$

— ✓ ✓ ✓ —

Example 2.8 Substitution may become handy even more often than with equations. Consider this inequality,

$$\frac{1}{5-\log x} + \frac{2}{\log x + 1} - 1 \le 0$$

As usual, we must start with assumptions,

$$\begin{cases} x > 0 \\ 5 - \log x \ne 0 \\ \log x + 1 \ne 0 \end{cases} \implies \begin{cases} x > 0 \\ \log x \ne 5 \\ \log x \ne -1 \end{cases} \implies \begin{cases} x > 0 \\ x \ne 10^5 \\ x \ne \frac{1}{10} \end{cases}$$

Let $\log x = t$, then

$$\begin{gathered} \frac{1}{5-t} + \frac{2}{t+1} - 1 \le 0 \\ \frac{t+1+2(5-t)-(5-t)(t+1)}{(5-t)(t+1)} \le 0 \\ \frac{t^2-5t+6}{(5-t)(t+1)} \le 0 \\ (t-2)(t-3)(5-t)(t+1) \le 0 \\ t \le -1 \quad \vee \quad 2 \le t \le 3 \quad \vee \quad t \le 5 \end{gathered}$$

We reverse the substitution and get four simple logarithmic inequalities,

$$\begin{gathered} \log x \le -1 \quad \vee \quad 2 \le \log x \le 3 \quad \vee \quad \log x \ge 5 \\ \log x \le \log 10^{-1} \quad \vee \quad \log 10^2 \le \log x \le \log 10^3 \quad \vee \quad \log x \ge \log 10^5 \end{gathered}$$

Because the logarithm with base 10 is an increasing function, and considering the assumptions above, we get

$$\boxed{0 < x < \frac{1}{10} \quad \vee \quad 100 \leq x \leq 1000 \quad \vee \quad x > 10^5}$$

—— ✓ ✓ ✓ ——

Example 2.9 For the last example let's take a combination of logarithmic and exponential inequalities,

$$\log_3 (2 \cdot 9^x + 16) - \log_3 (3^{x+1} - 6) \geq x$$

First the assumptions,

$$\begin{cases} 2 \cdot 9^x + 16 > 0 \\ 3^{x+1} - 6 > 0 \end{cases} \implies \begin{cases} x \in \mathbb{R} \\ 3^x > 2 \end{cases} \implies \quad 3^x > 3^{\log_3 2} \implies \quad x > \log_3 2$$

Now we combine the left-hand side into one logarithm, and rewrite the right-hand side as a log with the same base,

$$\log_3 \frac{2 \cdot 9^x + 16}{3^{x+1} - 6} \geq \log_3 3^x$$

Since logarithm with base 3 is an increasing function, and $3^x > 0$, we drop the logarithm to get,

$$\frac{2 \cdot 9^x + 16}{3^{x+1} - 6} \geq 3^x$$

Because the denominator is positive (by assumption), we can multiply both sides of the inequality by this denominator to get,

$$2 \cdot 9^x + 16 \geq 3^x(3^{x+1} - 6)$$

And we simplify,

$$\begin{gathered} 2 \cdot 9^x + 16 - 3^{2x+1} + 6 \cdot 3^x \geq 0 \\ 2 \cdot 3^{2x} + 16 - 3 \cdot 3^{2x} + 6 \cdot 3^x \geq 0 \\ 3^{2x} - 6 \cdot 3^x - 16 \leq 0 \end{gathered}$$

Let $3^x = t > 0$, then

$$\begin{gathered} t^2 - 6t - 16 \leq 0 \\ (t - 8)(t + 2) \leq 0 \\ -2 \leq t \leq 8 \quad \wedge \quad t > 0 \quad \implies \quad 0 < t \leq 8 \end{gathered}$$

Reversing the substitution we get

$$\begin{gathered} 3^x \leq 8 \\ \log_3 3^x \leq \log_3 8 \\ x \leq \log_3 8 \end{gathered}$$

Put it together with the initial assumption

$$x \leq \log_3 8 \quad \wedge \quad x > \log_3 2 \quad \implies \quad \boxed{\log_3 2 < x \leq \log_3 8}$$

—— ✓ ✓ ✓ ——

References

[1] *Matematyka – podstawy z elementami matematyki wyszej*, edited by B. Wikieł , PG publishing house, 2009.