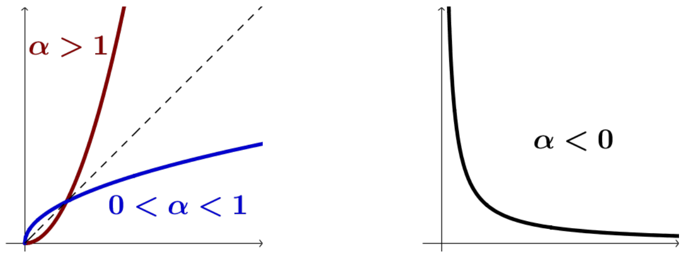
1 Power function and its graphs

Def. 1.1 The **power function** with exponent α is defined by

$$f(x) = x^{\alpha}$$

where $\alpha \in I\!R$, and the domain of f depends on the value of α.

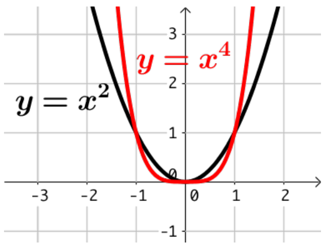
Since any positive number can be raised to any power, we can conclude that every power function is defined on $I\!R_+$. As far as graphs of power functions go, there are three basic shapes (not counting $f(x) = x$, which is a straight line) that a power function may have – it all depends on the exponent α.

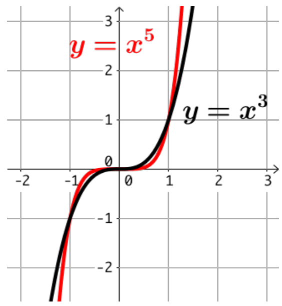


For some values of α, a power function may be defined on larger intervals, and the shape of the graph to the left of the $y-$axis will be similar to the one on the right, it will be either even or odd – the symmetry depends on whether the function takes on negative or positive values for negative x. Below we summarize all possible cases. In all of them $n, p \in \mathbb{N}$.

- $f(x) = x^n, \quad D_f = I\!R$

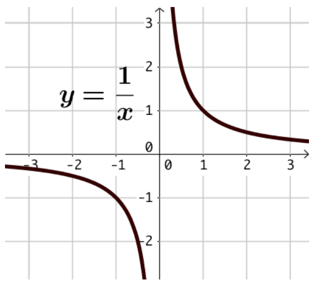
  These are monomials (polynomials consisting of only one term), so they exist for all $x \in I\!R$, and their graphs are even if n is even, and odd if n is odd. As powers increase the graphs get more flat on the interval $(-1, 1)$ and steeper on the outside of it.

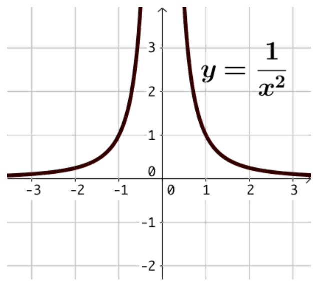




- $f(x) = x^{-n} = \dfrac{1}{x^n}, \quad D_f = \mathbb{R} - \{0\}$

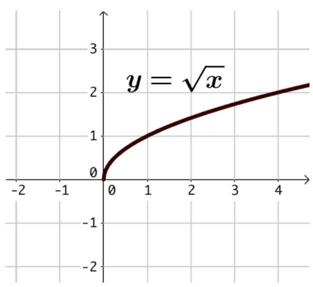
  Since there is division by x, zero is removed from the domain. If n is odd, the graph of the function is a hyperbola, if n is even it is symmetric about the $y-$axis.

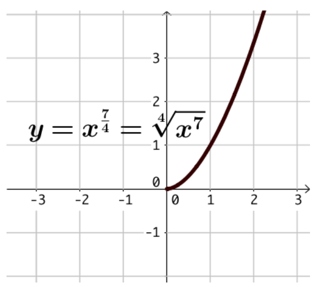


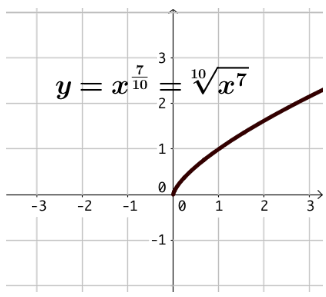


- $f(x) = x^{\frac{p}{n}}, \quad D_f = \mathbb{R}_+ \cup \{0\}$ for $n-$even,, $\quad D_f = \mathbb{R}$ for $n-$odd

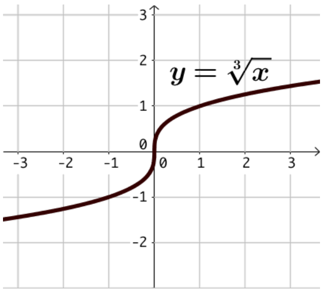
  Why the restriction on the domain for n even ? You may think of it this way,
  $x^{\frac{p}{n}} = \sqrt[n]{x}^p$ which is calculable provided the expression under the radical is nonnegative $(x \geq 0)$.

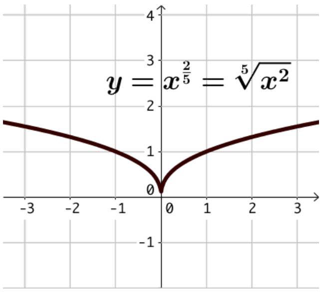


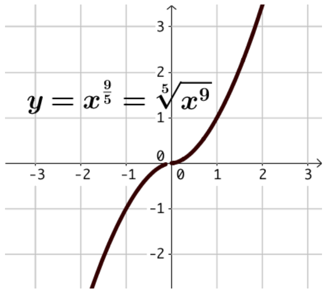




When n is odd, then there are no restriction on the sign of the expression under the radical. The value of p has the effect on the graph of the function. If p is even, then $x^p \geq 0$, so the values of the function are always nonnegative and the graph is even. If p is odd, then for negative x the value of the function is negative, and the graph of the function is odd.

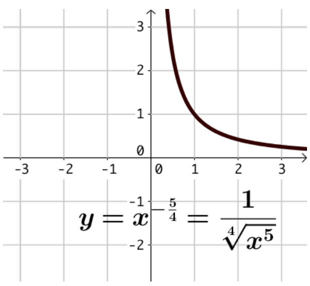


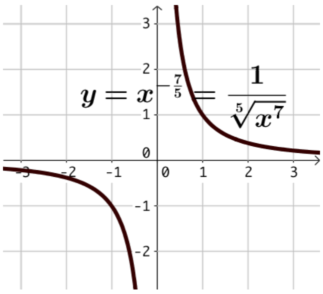


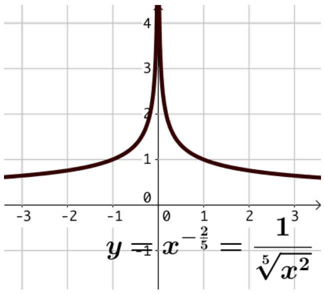


- $f(x) = x^{-\frac{p}{n}} = \dfrac{1}{\sqrt[n]{x^p}}$, $\quad D_f = \mathbb{R}_+$ for $n-$even, $\quad D_f = \mathbb{R} - \{0\}$ for $n-$odd

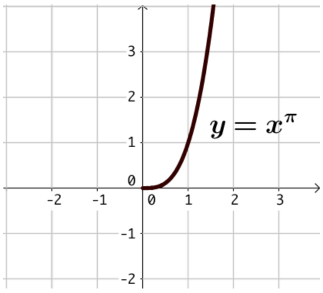
  Here we have a combination of the previous cases. On one hand we have division by x, so zero is out of the domain, on the other we also have radicals, so the rules for including $I\!R_-$ in the domain are the same as in the cases just above. Since the exponent is negative, the graphs have hyperbolic shape.

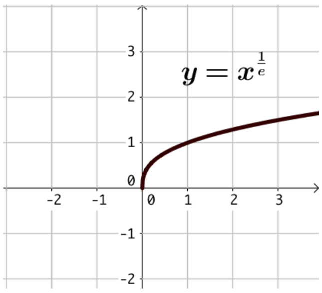


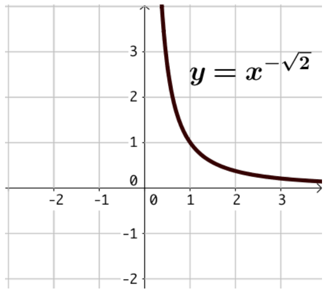




- $f(x) = x^0 = 1, \quad D_f = \mathbb{R} - \{0\}$.
- $f(x) = x^\alpha, \quad \alpha \in \mathbb{I}\mathbb{Q}, \quad D_f = \mathbb{R}_+$ for $\alpha < 0$, $\quad D_f = \mathbb{R}_+ \cup \{0\}$ for $\alpha > 0$.
  In case of irrational exponent x must not be negative, so we have one of three basic situations - depending on the size of the exponent α.







Why $x \geq 0$ in case of irrational exponents ? Well, how do we even evaluate something like $x^{\sqrt{3}}$. Let's see,

$$x^{\sqrt{3}} = x^{1.732\ldots} = x^1 \cdot x^{0.7} \cdot x^{0.03} \cdot x^{0.002} \cdots = x \cdot \sqrt[10]{x^7} \cdot \sqrt[100]{x^3} \cdot \sqrt[1000]{x^2} \ldots$$

Notice that if $x < 0$, then some of the terms in this expansion will not exist, e.g. $\sqrt[10]{x^7}$. Therefore, the domain of such functions does not include the negative values.

2 Radical equations and inequalities

When solving radical equations or inequalities the general rule is,

if you have $\sqrt[n]{\ldots}$ *term, take the* n^{th} *power of both sides*

But, be careful when n is even !!!

Why be careful ? Because when n is even, the power function is not one-to-one nor monotonic, so after applying (or removing) it the equality or the inequality will hold only for nonnegative arguments.

REMEMBER

$$\begin{aligned} a = b &\iff a^n = b^n \\ a < b &\iff a^n < b^n \\ a \leq b &\iff a^n \leq b^n \end{aligned}$$

- for $n-$even, $a, b \geq 0$
- for $n-$odd, $a, b \in \mathbb{R}$

EXAMPLE 2.1 Equations in which one side is a number do not require any extra assumptions.

- $\sqrt{x+7} = -3$
  This equation has no solution, since a square root never produces a negative number, so $\boxed{x \in \emptyset}$
- $\sqrt{x+8} = 5$
  First we must require that $x \geq -8$. Then we can square both sides of the equation (both sides are nonnegative) to get,
  $$x + 8 = 25 \quad \Longrightarrow \quad \boxed{x = 17}$$
- $\sqrt[3]{x-5} = -2$
  No assumptions are needed here, since a cubic root requires no restrictions on the argument. We take the third power of both sides, an operation which also doesn't require any assumptions,
  $$x - 5 = -8 \quad \Longrightarrow \quad \boxed{x = -3}$$

—— ✓ ✓ ✓ ——

EXAMPLE 2.2 Things get more complicated when both sides of the equation depend on x.

$$\sqrt{x-2} = 8 - x$$

We start with restrictions on x (the domain). The square root requires nonnegative argument, so

$$\text{assumption:} \quad x - 2 \geq 0 \quad \Rightarrow \quad x \geq 2$$

We want to get rid of the square root, so we square both sides of the equation. This operation requires both sides of the equation to be positive, so we have,

$$\text{another assumption :} \quad 8 - x \geq 0 \quad \Rightarrow \quad x \leq 8$$

After taking a square of both sides we get,

$$\begin{gathered} x - 2 = (8-x)^2 \\ x - 2 = 64 - 16x + x^2 \\ x^2 - 17x + 66 = 0 \\ (x-11)(x-6) = 0 \\ x = 6 \quad \vee \quad x = 11 \notin \langle 2, 8 \rangle \\ \boxed{x = 6} \end{gathered}$$

—— ✓ ✓ ✓ ——

Example 2.3 Sometimes taking a power has to be done more than once.

$$\sqrt{x+3} - \sqrt{2x-1} = 1$$

As usual, we must start with restrictions on x,

$$\text{assumptions:} \begin{cases} x+3 \geq 0 \\ 2x-1 \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq -3 \\ x \geq \frac{1}{2} \end{cases} \Rightarrow x \geq \frac{1}{2}$$

Now, to get rid of square roots we want to square the equation. To ensure that both sides are positive, we first move the second root to the right, and then square both sides

$$\begin{aligned} \sqrt{x+3} &= 1 + \sqrt{2x-1} \\ x+3 &= \left(1+\sqrt{2x-1}\right)^2 \\ x+3 &= 1 + 2\sqrt{2x-1} + 2x - 1 \\ 3 - x &= 2\sqrt{2x-1} \end{aligned}$$

We have to square the equation again, but to do it we must assume that both sides are nonnegative

$$\text{another assumption: } 3 - x \geq 0 \quad \Rightarrow \quad x \leq 3$$

And now we square,

$$\begin{gathered} 3 - x = 2\sqrt{2x-1} \\ 9 - 6x + x^2 = 4(2x-1) \\ x^2 - 14x + 13 = 0 \\ (x-1)(x-13) = 0 \\ x = 1 \quad \vee \quad x = 13 \notin \left\langle \frac{1}{2}, 3 \right\rangle \\ \boxed{x = 1} \end{gathered}$$

—— ✓ ✓ ✓ ——

As usual, things get more complicated when solving inequalities. We start with a few basic cases that show up when one solves more complex inequalities.

Example 2.4

- $\sqrt{x-3} \leq -4$

  Regardless of the assumption on x due to the square root, there is no solution to this inequality, since a square root always produces nonnegative numbers it cannot be smaller than -4. Thus,

  $$\boxed{x \in \emptyset}$$

- $\sqrt{x-3} \geq -4$

  When the direction of the inequality is reversed we have the opposite situation, a square root is always nonnegative so the inequality is satisfied as long as the root exists, so

  $$\boxed{x \geq 3}$$

- $\sqrt{x-3} < 2$

  First assumptions on x,

  $$\text{assumption: } x - 3 \geq 0 \quad \Longrightarrow \quad x \geq 3$$

When one side of the inequality is a positive number we can square it,

$$x - 3 < 4$$
$$x < 7 \quad \wedge \quad x \geq 3$$
$$\boxed{x \in \langle 3, 7)}$$

—— ✓ ✓ ✓ ——

Example 2.5 When both sides of the inequality contain x we have to consider two cases.

$$\sqrt{2x + 10} > 3x - 5$$

But first,

$$\text{assumption: } 2x + 10 \geq 0 \quad \Longrightarrow \quad x \geq -5$$

Now the two cases.

Case 1° The right-hand side is negative, that is

$$3x - 5 < 0 \quad \Rightarrow \quad x < \frac{5}{3}$$

In this case every x that satisfies the initial assumption will be a solution of the inequality, so

$$\text{solution of Case 1°: } x \in \left\langle -5, \frac{5}{3} \right)$$

Case 2° The right-hand side is nonnegative, that is

$$3x - 5 \geq 0 \quad \Rightarrow \quad x \geq \frac{5}{3}$$

We can square both sides of the inequality to get rid of the square root,

$$2x + 10 > (3x - 5)^2$$
$$2x + 10 > 9x^2 - 30x + 25$$
$$9x^2 - 32x + 15 < 0$$
$$\Delta = 484, \quad x = \frac{32 \pm 22}{18} = \frac{5}{9}, 3$$
$$x \in \left(\frac{5}{9}, 3\right) \quad \wedge \quad x \geq \frac{5}{3}$$
$$\text{solution of Case 2°: } x \in \left(\frac{5}{3}, 3\right)$$

Putting the solutions of Case 1° and Case 2° together we get,

$$\text{Case 1°} \cup \text{Case 2°} \ \Rightarrow \boxed{x \in \langle -5, 3)}$$

—— ✓ ✓ ✓ ——

Example 2.6 To avoid high order inequalities sometimes we may use substitution.

$$\sqrt{x^2 + x - 1} < 5x^2 + 5x - 9$$

There is a square root, so there may be restrictions on x,

$$\text{assummptions: } x^2 + x - 1 \geq 0 \quad \Leftrightarrow \quad x \leq \frac{-1 - \sqrt{5}}{2} \quad \vee \quad x \geq \frac{-1 + \sqrt{5}}{2}$$

We could now proceed as we did in the previous example, consider two cases and square both sides of the equation in the second case. However, this will leave us with a fourth order (!) polynomial (try it yourself).

To avoid it we may try to use a substitution. It is not always possible, but worth trying in a case like this one. To avoid high order polynomial we must avoid squaring, which means that we want to substitute for the entire square root. Thus, we try to rewrite the right-hand side of the inequality in the form of the expression under the root, possibly multiplied by a constant.

$$\sqrt{x^2+x-1} < 5x^2+5x-9 = 5(x^2+x-1)-4$$

Let $\sqrt{x^2+x-1} = t$, then our inequality becomes

$$t < 5t^2-4$$
$$5(t-1)\left(t+\frac{4}{5}\right) > 0$$
$$t < -\frac{4}{5} \quad \vee \quad t > 1$$

Having solved this inequality we reverse the substitution

$$\sqrt{x^2+x-1} < -\frac{4}{5} \quad \vee \quad \sqrt{x^2+x-1} > 1$$

The first inequality is of course never true, and the second one does not require any more assumption, we may simply square both sides of it,

$$x^2+x-1 > 1$$
$$x^2+x-2 > 0$$
$$(x+2)(x-1) > 0$$
$$x < -2 \quad \vee \quad x > 1$$

This answer has to be intersected with the initial assumptions, i.e. $x \leq \frac{-1-\sqrt{5}}{2} \quad \vee \quad x \geq \frac{-1+\sqrt{5}}{2}$.
And the final answer is $\boxed{x \in (-\infty,-2) \cup (1,\infty)}$.

—— ✓ ✓ ✓ ——

References

[1] *Matematyka – podstawy z elementami matematyki wyszej*, edited by B. Wikieł , PG publishing house, 2009.