Extremes of 2-variable functions

$$f(x,y) = x^3 + 3xy^2 - 51x - 24y$$

I PARTIAL DERIVATIVES:

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 51 \qquad \frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial x \partial y} = (6xy - 24)'_x = 6y$$

$$\frac{\partial f}{\partial y} = 6xy - 24 \qquad \frac{\partial^2 f}{\partial y^2} = 6x$$

II CRITICAL (= STATIONARY) POINTS

$$\begin{cases} 3x^2 + 3y^2 - 51 = 0 \quad / :3 \\ 6xy - 24 = 0 \quad / :6 \end{cases}$$

$$\begin{cases} x^2 + y^2 - 17 = 0 \\ xy - 4 = 0 \end{cases} \Rightarrow y = \frac{4}{x}$$

$$x^2 + \left(\frac{4}{x}\right)^2 - 17 = 0$$

$$x^2 + \frac{16}{x^2} - 17 = 0$$

$$\frac{x^4}{x^2} + \frac{16}{x^2} - 17 = 0 \quad / \cdot x^2$$

$$x^4 + 16 - 17x^2 = 0$$

$$x^4 - 17x^2 + 16 = 0$$

$$t = x^2, \quad t > 0$$

$$t^2 - 17t + 16 = 0$$

$$\Delta = b^2 - 4ac = 289 - 64 = 225$$

$$\sqrt{\Delta} = \sqrt{225} = 15$$

$$t_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{17 - 15}{2} = \frac{2}{2} = 1 \in D_t$$

$$t_2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{17 + 15}{2} = \frac{32}{2} = 16 \in D_t$$

$$x^2 = 1 \quad \vee \quad x^2 = 16$$

$$x_1 = 1 \ \vee \ x_2 = -1 \ \vee \ x_3 = 4 \ \vee \ x_4 = -4$$

$$P_1 = (1, 4) \qquad P_2 = (-1, -4) \qquad P_3 = (4, 1) \qquad P_4 = (-4, -1)$$

III DETERMINANTS

$$D_f = \begin{vmatrix} 6x & 6y \\ 6y & 6x \end{vmatrix} = 36x^2 - 36y^2$$

$D_f(1,4) = 36 \cdot 1^2 - 36 \cdot 4^2 < 0$, saddle point

$D_f(-1,-4) = 36 \cdot (-1)^2 - 36 \cdot (-4)^2 < 0$, saddle point

$D_f(4,1) = 36 \cdot (4)^2 - 36 \cdot (1)^2 > 0$, $(6x)\Big|_{(4,1)} = 24 > 0$ MINIMUM

$D_f(-4,-1) = 36 \cdot (-4)^2 - 36 \cdot (-1)^2 > 0$, $(6x)\Big|_{(-4,-1)} = -24 < 0$ MAXIMUM

IV ANSWER

$f(4,1) = 4^3 + 3 \cdot 4 \cdot 1^2 - 51 \cdot 4 - 24 \cdot 1 = -152$ is a minimum

$f(-4,-1) = (-4)^3 + 3 \cdot (-4) \cdot (-1)^2 - 51 \cdot (-4) - 24 \cdot (-1) = -64 - 12 + 204 + 24 =$

$= 152$ is a maximum

SADDLE POINTS:

$f(1,4) = 1^3 + 3 \cdot 1 \cdot 4^2 - 51 \cdot 1 - 24 \cdot 4 = 1 + 48 - 51 - 96 = -98$

$P = (1, 4, -98)$

$f(-1,-4) = (-1)^3 + 3 \cdot (-1) \cdot (-4)^2 - 51 \cdot (-1) - 24 \cdot (-4) = -1 - 48 + 51 + 96 = 98$

$P = (-1, -4, 98)$

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On this page I present two graphs of f(x, y) in a close surrounding of points (4,1) and (-4,-1). The surrounding of (4,1) here is [3.9,4,1]x[0.9,1.1] and the surrounding of (-4,-1) is [-4.1,-3.9]x[-1.1,-0.9].