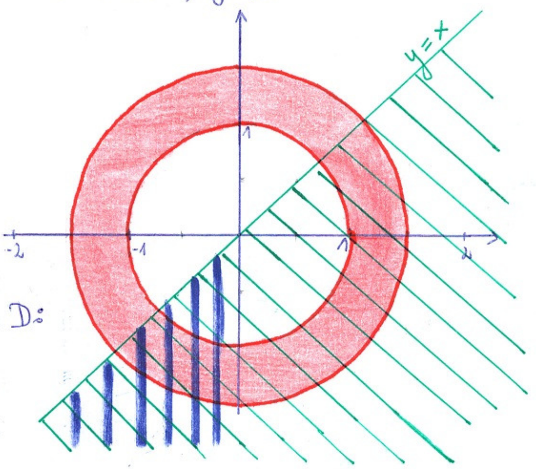
An example of using polar coordinates in double integrals:

Exercise: Calculate $\iint\limits_D \frac{1}{(1-x^2-y^2)^2}\,dx\,dy$, where $D=\{(x,y): 1\leqslant x^2+y^2\leqslant \frac{9}{4},$ and $x\leqslant 0$ and $y\leqslant x\}$

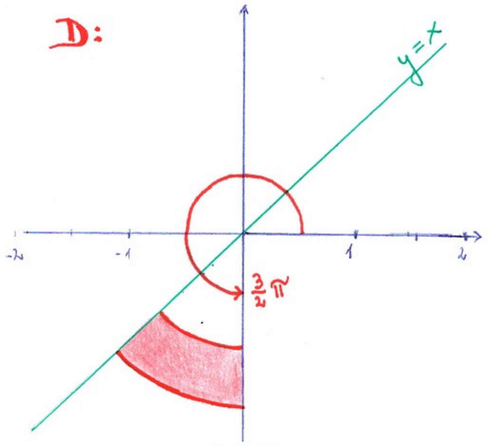
SOLUTION: First, I draw a region D:

* $1\leqslant x^2+y^2\leqslant \left(\frac{3}{2}\right)^2$

* $x\leqslant 0$, $y\leqslant x$ *



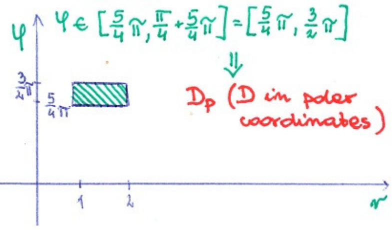
$\Rightarrow$



$r\in[1,\frac{3}{2}]$

$\varphi\in[\frac{5}{4}\pi,\frac{\pi}{4}+\frac{5}{4}\pi]=[\frac{5}{4}\pi,\frac{3}{2}\pi]$

$\Downarrow$



D_p (D in polar coordinates)

I will use the formula:

$$\iint\limits_D f(x,y)\,dx\,dy=\iint\limits_{D_p} f(r\cos\varphi, r\sin\varphi)\cdot\underbrace{r}_{\text{the Jacobian!}}\,dr\,d\varphi$$

$$f(x,y)=\frac{1}{(1-x^2-y^2)^2}$$

$$f(r\cos\varphi, r\sin\varphi)=\frac{1}{(1-r^2\cos^2\varphi-r^2\sin^2\varphi)^2}=\frac{1}{(1-r^2(\cos^2\varphi+\sin^2\varphi))^2}=\frac{1}{(1-r^2)^2}$$

$$\iint\limits_D\left(\frac{1}{(1-x^2-y^2)^2}\right)dx\,dy=\int\limits_{\varphi=\frac{5}{4}\pi}^{\frac{3}{2}\pi}\int\limits_{r=1}^{\frac{3}{2}}\left(\frac{1}{(1-r^2)^2}\right)\cdot r\,dr\,d\varphi=$$

$$=\int\limits_{\frac{5}{\pi}\pi}^{\frac{3}{2}\pi}\left(\int\limits_{1}^{3/2}\left(\frac{r}{(1-r^2)^2}\right)dr\right)d\varphi=\left\{\begin{array}{l}1-r^2=t\\ -2r\,dr=dt\\ dr=\frac{dt}{-2r}\\ \begin{array}{c|c|c} r^2 & 1 & \frac{3}{2}\\ \hline 1-r^2 & 0 & -\frac{5}{4}\end{array}\end{array}\right\}=\int\limits_{\frac{5}{4}\pi}^{\frac{3}{2}\pi}\left(-\frac{1}{2}\int\limits_{0}^{-5/4}\left(\frac{1}{t^2}\right)dt\right)d\varphi=$$

$$=\int\limits_{\frac{5\pi}{4}}^{\frac{3\pi}{2}}d\varphi\cdot\underbrace{\int\limits_{0}^{-\frac{5}{4}}\frac{1}{t^2}\,dt}_{0\notin D\frac{1}{t^2}}$$

$$\lim_{A\to 0}\int\limits_A^{-\frac{5}{4}}t^{-2}\,dt=\lim_{A\to 0}\frac{t^{-1}}{-1}\Bigg|_A^{-\frac{5}{4}}=\lim_{A\to 0}-\left(\frac{1}{t}\right)\Bigg|_A^{-\frac{5}{4}}=\lim_{A\to 0}-\left(\frac{1}{-5/4}-\frac{1}{A}\right)=\lim_{A\to 0}\left(\frac{4}{5}+\underset{\frac{1}{0}=\infty}{\frac{1}{A}}\right)=\underline{\underline{\infty}}$$

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