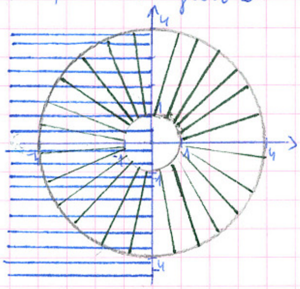
An example of using polar coordinates in double integrals

Exercise: Calculate $\iint_D (x \cdot y)\, dx\,dy$, where $D = \{ (x,y) : 1 \leq x^2 + y^2 \leq 16 \text{ and } x \leq 0 \}$

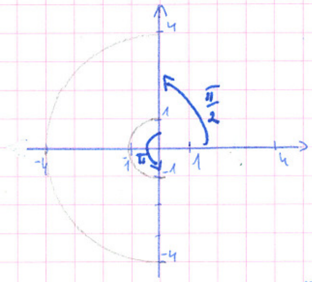
Solution: First, I drew region D:

* $1 \leq x^2 + y^2 \leq 4^2$
* $x \leq 0$



$\Rightarrow$

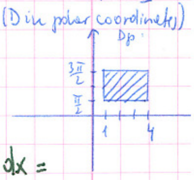
D:



$r \in [1, 4]$

$\varphi \in \left[\frac{\pi}{2}; \frac{\pi}{2} + \pi\right] = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

(D in polar coordinates)



I will use the formula: $\iint_D f(x,y)\, dx\,dy = \iint_{D_p} f(r\cos\varphi, r\sin\varphi) \cdot r\, dr\,d\varphi$

the Jacobian

$f(x,y) = x \cdot y$

$f(r\cos\varphi, r\sin\varphi) = r \cdot \cos\varphi \cdot r \cdot \sin\varphi = r^2 \cos\varphi \sin\varphi$

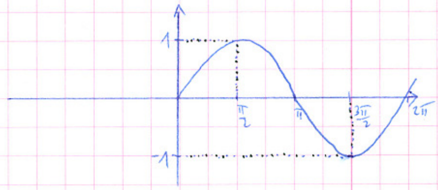
$\iint_D (x \cdot y)\, dx\,dy = \int_{\varphi=\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{r=1}^{4} (r^2 \cos\varphi \sin\varphi) \cdot r\, dr\,d\varphi = \int_{r=1}^{4} r^3\, dr \cdot \int_{\varphi=\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\varphi \sin\varphi\, d\varphi =$

$\int f^n(x) \cdot f'(x)\, dx = \frac{f^{n+1}(x)}{n+1} + C$

$= \frac{r^4}{4}\Big|_{r=1}^{4} \cdot \frac{\sin^2\varphi}{2}\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \left(64 - \frac{1}{4}\right) \cdot \frac{1}{2}\left(\sin\varphi \cdot \sin\varphi\right)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} =$

$= 63\frac{3}{4} \cdot \frac{1}{2}\Big[\underbrace{\Big(\underbrace{\sin\frac{3\pi}{2}}_{-1} \cdot \underbrace{\sin\frac{3\pi}{2}}_{-1}\Big)}_{1} - \underbrace{\Big(\underbrace{\sin\frac{\pi}{2}}_{1} \cdot \underbrace{\sin\frac{\pi}{2}}_{1}\Big)}_{1}\Big] =$

$= 63\frac{3}{8} \cdot \underbrace{(1 - 1)}_{0} = \underline{0}$



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