An example of using polare coordinates in double integrals

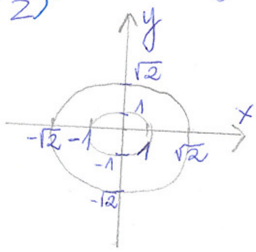
Exercise: Calculate $\iint_D \left(\sqrt[3]{x^2+y^2}\right) dx\,dy$, where

$$D = \{(x,y) : 1 \leqslant x^2+y^2 \leqslant 2 \text{ and } y \geqslant 0\}$$

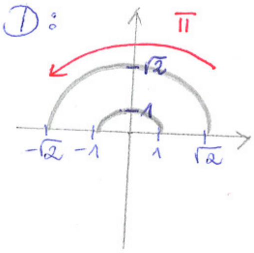
Solution: First I drew region D:

$1 \leqslant x^2+y^2 \leqslant (\sqrt{2})^2$

$y \geqslant 0$



D:

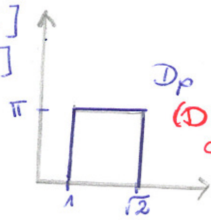


I will use the formula:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_p} f(r\cos\varphi, r\sin\varphi)\cdot \underbrace{r}_{\text{the Jacobian !}}\,dr\,d\varphi$$

$r \in [1, \sqrt{2}]$

$\varphi \in [0, \pi]$



D_p (D in polare coordinates)

$f(x,y) = \sqrt[3]{x^2+y^2}$

$$f(r\cos\varphi, r\sin\varphi) = \sqrt[3]{r^2\cos^2\varphi + r^2\sin^2\varphi} = \sqrt[3]{r^2}\cdot\sqrt{\underbrace{\cos^2\varphi+\sin^2\varphi}_{1}} = r^{\frac{2}{3}}$$

$$\iint_D \left(\sqrt[3]{x^2+y^2}\right)dx\,dy = \int_{\varphi=0}^{\pi}\int_{r=1}^{\sqrt{2}} r^{\frac{2}{3}} \cdot r \; dr\,d\varphi =$$

the Jacobian

$$\int_0^{\pi}\left(\int_1^{\sqrt{2}}\left(r^{1\frac{2}{3}}\right)dr\right)d\varphi = \int_0^{\pi}\left(\frac{r^{2\frac{2}{3}}}{2\frac{2}{3}}\right)\Bigg|_1^{\sqrt{2}} d\varphi =$$

$$\int_0^{\pi}\left(\frac{\sqrt{2}^{\frac{8}{3}}}{\frac{8}{3}} - \frac{1^{8/3}}{8/3}\right)d\varphi = \int_0^{\pi}\frac{3}{8}\left(\sqrt[3]{16}-1\right)d\varphi =$$

$$\frac{3}{8}\left(\sqrt[3]{16}-1\right)\cdot\varphi\,\Big|_0^{\pi} = \underline{\frac{3}{8}\left(\sqrt[3]{16}-1\right)\cdot\pi}$$

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