Iwona Markiewicz

Exercise: Establish the convergence (none, absolute, conditional) of:

$$\sum_{1}^{\infty} (-1)^n \cdot \left(\frac{n-1}{n+1}\right)^{n^2}$$

$$a_n = \left(\frac{n-1}{n+1}\right)^{n^2}$$

Summation, from 1 to infinity, of terms: minus 1 to the n-th power times n minus 1 over n plus one to power square. It's an alternating series, so I will consider a_n to be only n minus 1 over n plus one to power n square (without $(-1)^n$ part)

We need to perform alternating series test, which involves checking 3 conditions:

1° $a_n > 0$?

$$\left(\frac{n-1}{n+1}\right)^{n^2} > 0$$

1° Is a_n greater than zero for all $n \geq 1$?
In the nominator we have always smaller number then in the denominator, whose number is greater. So the fraction in bracket will be always smaller then 1 and all numbers p such that $|p| < 1$ tend to zero when raised to n^2.

2° $\{a_n\} \downarrow$?

$$\left(\frac{n-1}{n+1}\right)^{n^2} \downarrow$$

2° Is a_n a decreasing sequence?
We can see, that when we divide smaller number over larger, and rise it to the power n square, we will have smaller fraction. So the sequence decreases.

3° $\lim\limits_{n\to\infty} a_n = 0$?

$$\lim_{n\to\infty}\left(\frac{n+1-2}{n+1}\right)^{n^2} = \lim_{n\to\infty} \overset{\substack{e\\ \uparrow}}{\boxed{\left(1+\frac{1}{\frac{n+1}{-2}}\right)^{\frac{n+1}{-2}}}} \cdot \frac{-2}{n+1}\cdot n^2 =$$

$$\lim_{n\to\infty} e^{\left(\frac{-2n^2}{n+1}\right)} = \lim_{n\to\infty} e^{\left(\frac{-2n^{\cancel{2}}}{\cancel{n}(1+\underset{\overset{\parallel}{0}}{\boxed{\frac{1}{n}}})}\right)} = e^{\boxed{-2n} \to -\infty} = e^{-\infty} = 0$$

3° Is the limit of a_n equal to 0 ?
We can add 1 and subtract -2 instead of writing -1 in the nominator. Then we have to seperate two expressions and we get 1 and $\frac{-2}{n+1}$, what we can avert and get $\frac{1}{\frac{n+1}{-2}}$.
At the same time, we insert in the power two reverse expressions $\frac{n+1}{-2} \cdot \frac{-2}{n+1}$, what gives us 1. Now we see what tends to e. We extract n in front of the bracket in the denominator and we get e tending to $-\infty$, what is 0.

So the series is **convergent**.
Now we have to check limit of a_n by different test. I choose **root test**.

$$\lim_{n\to\infty}\sqrt[n]{a_n} = q$$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n^2}} = \left(\left(\frac{n-1}{n+1}\right)^{n^{\cancel{2}}}\right)^{\frac{1}{\cancel{n}}} =$$

$$= \left(\frac{n-1}{n+1}\right)^n = \left(\frac{n+1-2}{n+1}\right)^n = \left(1-\frac{2}{n+1}\right)^n =$$

$$= \boxed{\left(1+\frac{1}{\frac{n+1}{-2}}\right)^{\frac{n+1}{-2}}} \to e \cdot \frac{(-2)}{n+1}\cdot n = e^{\frac{-2n}{n+1}}$$

$$\lim_{n\to\infty} e^{\left(\frac{-2\cancel{n}}{\cancel{n}(1+\underset{\overset{\parallel}{0}}{\boxed{\frac{1}{n}}})}\right)} = e^{-2} = \frac{1}{e^2} < 1$$

At first we take a_n to the power of one half. Then we can reduce n. Now we do everything like in last example (3°).

Our q is smaller then 1, so the series is convergent.
Now we know that the series is
absolutely convergent!