What to say in front of the blackboard: – a brief tutorial

Exercise: Determine whether the series is divergent, absolutely convergent or conditionally convergent:

$$\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$$

Summation from 1 to infinity of terms minus 100 to the n-th power divided by n factorial

Answer: $$\sum_{n=1}^{\infty} \frac{(-100)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 100^n}{n!}$$

First of all, we can change our series to the alternating series form. Hence, we put -1 to the n-th power in front of numerator using the formula $(a \cdot b)^n = a^n \cdot b^n$

Thereby, we have alternating series and we consider a_n as:

$$a_n = \frac{100^n}{n!} \quad \text{(without the } (-1)^n \text{ part)}$$

We need to carry out the alternating series test. It consists of three conditions, which must be satisfied.

1° $\forall_n \; a_n > 0$

For each $n \geqslant 1$ a_n has to be greater than zero.

$$a_n = \frac{100^n}{n!}$$

Condition $\frac{100^n}{n!}$ is greater than zero is met, because n assumes only greater or equal than 1 values.

2° $\forall_n \; a_n \geqslant a_{n+1}$

Secondly, we have to check whether a_n is greater or equal than a_{n+1}. (Is it decreasing? $\Rightarrow \{a_n\}\downarrow$)

$$\frac{a_{n+1}}{a_n} = \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \frac{100}{n+1}$$

We can notice that $a_n \geqslant a_{n+1}$ if and only if $\frac{a_{n+1}}{a_n} < 1$.

$$\frac{100}{n+1} < 1 \iff n+1 > 100 \Rightarrow$$

$$\Rightarrow n > 99$$

Therefore, from appropriate inequality we conclude that the series start to decrease, when n assumes values greater than 99. Hence the 2nd condition is fulfilled

3° $\lim_{n \to \infty} a_n = 0$

Finally, in the third condition we need to check whether the limit of a_n is zero

$$\lim_{n \to \infty} \frac{100^n}{n!} = 0$$

The third condition is also satisfied, because $n!$ begins to grow faster than 100^n from the moment when $n \geqslant 100$. So the limit of a_n equals 0.

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 100^n}{n!}$ is convergent

All three conditions are fulfilled, thus it isn't divergent. The alternating series test proved that it is convergent

Now we have to check convergence of a given series:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot 100^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{100^n}{n!}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = q$$

Here a_n is the same as in alternating series test, but in this case it is not alternating series, thus we will use other suitable method. In this example convergence may be verified by the ratio test.

By the ratio test we have:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} =$$

$$= \lim_{n \to \infty} \frac{\cancel{100^n} \cdot 100 \cdot \cancel{n!}}{\cancel{n!} \cdot (n+1) \cdot \cancel{100^n}} = \lim_{n \to \infty} \underbrace{\frac{100}{n+1}}_{\frac{100}{\infty}} = 0 = q$$

If $q < 1$, the series converges

Conclusion:

If $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ is convergent and $\sum_{n=1}^{\infty} \left| \frac{(-100)^n}{n!} \right|$ is also convergent, then:

$$\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$$ is said tobe absolutely convergent

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